

# Cubature Methods and Applications to Option Pricing

by

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# Declaration

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# Abstract

In this thesis, higher order numerical methods for weak approximation of solutions of stochastic differential equations (SDEs) are presented. They are motivated by option pricing problems in finance where the price of a given option can be written as the expectation of a functional of a diffusion process. Numerical methods of order at most one have been the most used so far and higher order methods have been difficult to perform because of the unknown density of iterated integrals of the  $d$ -dimensional Brownian motion present in the stochastic Taylor expansion. In 2001, Kusuoka constructed a higher order approximation scheme based on Malliavin calculus. The iterated stochastic integrals are replaced by a family of finitely-valued random variables whose moments up to a certain fixed order are equivalent to moments of iterated Stratonovich integrals of Brownian motion. This method has been shown to outperform the traditional Euler-Maruyama method. In 2004, this method was refined by Lyons and Victoir into Cubature on Wiener space. Lyons and Victoir extended the classical cubature method for approximating integrals in finite dimension to approximating integrals in infinite dimensional Wiener space. Since then, many authors have intensively applied these ideas and the topic is today an active domain of research. Our work is essentially based on the recently developed higher order schemes based on ideas of the Kusuoka approximation and Lyons-Victoir “Cubature on Wiener space” and mostly applied to option pricing. These are the Ninomiya-Victoir (N-V) and Ninomiya-Ninomiya (N-N) approximation schemes. It should be stressed here that many other applications of these schemes have been developed among which is the Alfonsi scheme for the CIR process and the decomposition method presented by Kohatsu and Tanaka for jump driven SDEs.

After sketching the main ideas of numerical approximation methods in Chapter 1, we start Chapter 2 by setting up some essential terminologies and definitions. A discussion on the stochastic Taylor expansion based on iterated Stratonovich integrals is presented, we close this chapter by illustrating this expansion with the Euler-Maruyama approximation scheme. Chapter 3 contains the main ideas of Kusuoka approximation scheme, we concentrate on the implementation of the algorithm. This scheme is applied to the pricing of an Asian call option and numerical results are presented. We start Chapter 4 by taking a look at the classical cubature formulas after which we propose in

a simple way the general ideas of “Cubature on Wiener space” also known as the Lyons-Victoir approximation scheme. This is an extension of the classical cubature method. The aim of this scheme is to construct cubature formulas for approximating integrals defined on Wiener space and consequently, to develop higher order numerical schemes. It is based on the stochastic Stratonovich expansion and can be viewed as an extension of the Kusuoka scheme. Applying the ideas of the Kusuoka and Lyons-Victoir approximation schemes, Ninomiya-Victoir and Ninomiya-Ninomiya developed new numerical schemes of order 2, where they transformed the problem of solving SDE into a problem of solving ordinary differential equations (ODEs). In Chapter 5, we begin by a general presentation of the N-V algorithm. We then apply this algorithm to the pricing of an Asian call option and we also consider the optimal portfolio strategies problem introduced by Fukaya. The implementation and numerical simulation of the algorithm for these problems are performed. We find that the N-V algorithm performs significantly faster than the traditional Euler-Maruyama method. Finally, the N-N approximation method is introduced. The idea behind this scheme is to construct an ODE-valued random variable whose average approximates the solution of a given SDE. The Runge-Kutta method for ODEs is then applied to the ODE drawn from the random variable and a linear operator is constructed. We derive the general expression for the constructed operator and apply the algorithm to the pricing of an Asian call option under the Heston volatility model.

# Opsomming

In hierdie proefskrif, word 'n hoërorde numeriese metode vir die swak benadering van oplossings tot stogastiese differensiaalvergelykings (SDV) aangebied. Die motivering vir hierdie werk word gegee deur 'n probleem in finansies, naamlik om opsiepryse vas te stel, waar die prys van 'n gegewe opsie beskryf kan word as die verwagte waarde van 'n funksionaal van 'n diffusie proses. Numeriese metodes van orde, op die meeste een, is tot dus ver in algemene gebruik. Dit is moeilik om hoërorde metodes toe te pas as gevolg van die onbekende digtheid van herhaalde integrale van d-dimensionele Brown-beweging teenwoordig in die stogastiese Taylor ontwikkeling. In 2001 het Kusuoka 'n hoërorde benaderings skema gekonstrueer wat gebaseer is op Malliavin calculus. Die herhaalde stogastiese integrale word vervang deur 'n familie van stogastiese veranderlikes met eindige waardes, wat se momente tot 'n sekere vaste orde bestaan. Dit is al gedemonstreer dat hierdie metode die tradisionele Euler-Maruyama metode oortref. In 2004 is hierdie metode verfyn deur Lyons en Victoir na volumeberekening op Wiener ruimtes. Lyons en Victoir het uitgebrei op die klassieke volumeberekening metode om integrale te benader in eindige dimensie na die benadering van integrale in oneindige dimensionele Wiener ruimte. Sedertdien het menige outeurs dié idees intensief toegepas en is die onderwerp vandag 'n aktiewe navorsings gebied. Ons werk is hoofsaaklik gebaseer op die onlangse ontwikkeling van hoërorde skemas, wat op hul beurt gebaseer is op die idees van Kusuoka benadering en Lyons-Victoir "Volumeberekening op Wiener ruimte". Die werk word veral toegepas op die prysvastelling van opsies, naamlik Ninomiya-Victoir en Ninomiya-Ninomiya benaderings skemas. Dit moet hier beklemtoon word dat baie ander toepassings van hierdie skemas al ontwikkel is, onder meer die Alfonsi skema vir die CIR proses en die ontbinding metode wat voorgestel is deur Kohatsu en Tanaka vir sprong aangedrewe SDVs. Na 'n skets van die hoof idees agter metodes van numeriese benadering in Hoofstuk 1, begin Hoofstuk 2 met die neersetting van noodsaaklike terminologie en definisies. 'n Diskussie oor die stogastiese Taylor ontwikkeling, gebaseer op herhaalde Stratonovich integrale word uiteengeset, waarna die hoofstuk afsluit met 'n illustrasie van dié ontwikkeling met die Euler-Maruyama benaderings skema. Hoofstuk 3 bevat die hoofgedagtes agter die Kusuoka benaderings skema, waar daar ook op die implementering van die algoritme gekonsentreer word. Hierdie skema is van toepassing op die prysvastelling van 'n Asiatiese

call-opsie, numeriese resultate word ook aangebied. Ons begin Hoofstuk 4 deur te kyk na klassieke volumeberekennings formules waarna ons op 'n eenvoudige wyse die algemene idees van "Volumeberekening op Wiener ruimtes", ook bekend as die Lyons-Victoir benaderings skema, as 'n uitbreiding van die klassieke volumeberekening metode gebruik. Die doel van hierdie skema is om volumeberekening formules op te stel vir benaderings integrale wat gedefinieer is op Wiener ruimtes en gevolglik, hoërde numeriese skemas te ontwikkel. Dit is gebaseer op die stogastiese Stratonovich ontwikkeling en kan beskou word as 'n ontwikkeling van die Kusuoka skema. Deur Kusuoka en Lyon-Victoir se idees oor benaderings skemas toe te pas, het Ninomiya-Victoir en Ninomiya-Ninomiya nuwe numeriese skemas van orde 2 ontwikkel, waar hulle die probleem omgeskakel het van een waar SDVs opgelos moet word, na een waar gewone differensiaalvergelykings (GDV) opgelos moet word. Hierdie twee skemas word in Hoofstuk 5 uiteengeset. Alhoewel die benaderings soortgelyk is, is daar 'n beduidende verskil in die algoritmes self. Hierdie hoofstuk begin met 'n algemene uiteensetting van die Ninomiya-Victoir algoritme waar 'n arbitrêre vaste tyd horison,  $T$ , gebruik word. Dié word toegepas op opsieprysvastelling en optimale portefeulje strategie probleme. Verder word numeriese simulaties uitgevoer, die prestasie van die Ninomiya-Victoir algoritme was bestudeer en vergelyk met die Euler-Maruyama metode. Ons maak die opmerking dat die Ninomiya-Victoir algoritme aansienlik vinniger is. Die belangrikste resultaat van die Ninomiya-Ninomiya benaderings skema word ook voorgestel. Deur die idee van 'n Lie algebra te gebruik, het Ninomiya en Ninomiya 'n stogastiese veranderlike met GDV-waardes gekonstrueer wat se gemiddeld die oplossing van 'n gegewe SDV benader. Die Runge-Kutta metode vir GDVs word dan toegepas op die GDV wat getrek is uit die stogastiese veranderlike en 'n lineêre operator gekonstrueer. 'n Veralgemeende uitdrukking vir die gekonstrueerde operator is afgelei en die algoritme is toegepas op die prysvasstelling van 'n Asiatiese opsie onder die Heston onbestendigheds model.

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# Chapter 1

## Introduction

Stochastic Differential Equations (SDEs) are equations obtained by allowing randomness in the coefficients of differential equations. They provide powerful models for a multitude of phenomena and processes encountered in a wide variety of disciplines. In filtering, they permit one to build models which help to filter the noise from some observations. In numerical analysis, they help to solve parabolic or elliptic Partial Differential Equations in situations where the deterministic algorithms become difficult or inefficient to use. In optimal stopping, one problem is to find a stopping strategy that gives the best result in long run. In Mathematical Finance, they are important tools in the modelling of risky securities, most notably in option pricing. As differential equations, since the class of stochastic differential equations that admits explicit solutions is rather limited, it is crucial to construct fast, accurate and robust algorithms for their numerical approximation.

Let  $B = (B^1, \dots, B^d)$  be a  $d$ -dimensional standard Brownian motion defined on the filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , which is assumed to satisfy the usual conditions.

Let  $\bar{V}_0, V_1, \dots, V_d \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ , where  $C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$  denotes the space of  $\mathbb{R}^N$ -valued infinitely differentiable functions defined on  $\mathbb{R}^N$  whose derivatives of any order are bounded. Each element of  $C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$  can be viewed as a differential operator through Remark 1.0.3 below. We consider the corresponding Itô stochastic differential equations (SDEs)

$$dX_t^x = \bar{V}_0(X_t^x) dt + \sum_{j=1}^d V_j(X_t^x) \cdot dB_t^j \quad (1.0.1)$$

with initial value  $X_0^x = x \in \mathbb{R}^N$ .

Throughout this thesis we will refer to a stochastic process  $X$  by the notation  $(X_t^x)_{t \geq 0}$  or  $(X(t, x))_{t \geq 0}$ .

**Definition 1.0.1.** We define a (strong) solution of the SDE (1.0.1) as an  $\mathbb{F}$ -adapted stochastic process  $(X(t, x))_{t \geq 0}$  with continuous paths such that

$$X(t, x) = x + \int_0^t \bar{V}_0(X_s^x) ds + \sum_{j=1}^d \int_0^t V_j(X_s^x) \cdot dB_s^j, \quad (1.0.2)$$

for all  $t$ .

**Remark 1.0.2.** The Lebesgue and the Itô integrals in the above definition have to be well-defined, e.g. we may require that

$$\mathbb{E} \left( \int_0^t \left( \|\bar{V}_0(X_s^x)\| + \sum_{j=1}^d \|V_j(X_s^x)\|^2 \right) ds \right) < \infty$$

for all  $t$ , where  $\|\cdot\|$  denotes the euclidean norm.

We will often work on a finite time horizon  $T > 0$  and in this case, all considered processes are defined on  $[0, T]$ . By the solution to a SDE, we will always understand a strong solution as defined above.

Even though Definition 1.0.1 considers a SDE in terms of the Itô stochastic integral, it is useful to reformulate it in terms of the Stratonovich integral as explained here. For this we introduce the smooth map  $V_0 : \mathbb{R}^N \rightarrow \mathbb{R}^N$  defined in a compact notation by

$$V_0 = \bar{V}_0 - \frac{1}{2} \sum_{j=1}^d DV_j(x) \cdot V_j(x) \quad (1.0.3)$$

where

$$DV_j = \begin{pmatrix} \frac{\partial V_j^1}{\partial x_1} & \frac{\partial V_j^1}{\partial x_2} & \cdots & \frac{\partial V_j^1}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial V_j^N}{\partial x_1} & \frac{\partial V_j^N}{\partial x_2} & \cdots & \frac{\partial V_j^N}{\partial x_N} \end{pmatrix}$$

and  $V_j^k \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$  is the  $k$ -component of  $V_j$  for  $k = 1, \dots, N$ . More precisely, we have

$$V_0^i = \bar{V}_0^i - \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^N V_j^k \frac{\partial V_j^i}{\partial x_k}, \quad (1.0.4)$$

for  $i = 1, 2, \dots, N$ .

A (strong) solution of the following SDE in Stratonovich form,

$$dX(t, x) = V_0(X_t^x) dt + \sum_{j=1}^d V_j(X_t^x) \circ dB_t^j, \quad (1.0.5)$$

with initial condition  $X(0, x) = x$  is the process  $(X(t, x))_t$  which satisfies the integral equation

$$X(t, x) = x + \int_0^t V_0(X_s^x) ds + \sum_{j=1}^d \int_0^t V_j(X_s^x) \circ dB_s^j. \quad (1.0.6)$$

In the above equations, “ $\circ$ ” denotes the Stratonovich integral (see Definition 2.2.1 for more details). In this thesis, we are mainly concerned with SDEs in Stratonovich form, except for a few cases where we use the Itô form.

**Remark 1.0.3.** A vector field  $V \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$  can be identified with a first-order differential operator via

$$Vf(x) = \sum_{j=1}^N V^j(x) \frac{\partial f}{\partial x_j}(x), \quad \text{for } f \in C_b^\infty(\mathbb{R}^N; \mathbb{R}), \quad (1.0.7)$$

where  $V^j \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$  is the  $j$ -component of  $V$  for  $j = 1, \dots, N$ .

To point out the connection between SDEs and Partial Differential Equations (PDEs), we introduce the second-order differential operator

$$Lf(x) = V_0f(x) + \frac{1}{2} \sum_{i=1}^d V_i^2 f(x), \quad x \in \mathbb{R}^N \quad (1.0.8)$$

where  $V_i^2 f(x) = V_i(V_i f)(x)$ . Let us consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) &= Lu(t, x) \\ u(0, x) &= f(x) \end{cases}, \quad (1.0.9)$$

where  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is a given function and the operator  $L$  is understood to act on the  $x$ -variable of  $u$  only. The classical Feynman-Kac Formula gives a probabilistic representation of the solution  $u$  of Equation (1.0.9). More precisely, we have the following:

**Proposition 1.0.4.** (Feynman-Kac Formula). Under appropriate regularity conditions on the vector fields and on  $f$ , the solution of the heat equation (1.0.9) is given by

$$u(t, x) = \mathbb{E}(f(X_t^x)).$$

We also introduce the semi-group of linear operators  $\{P_t\}_{t \in [0, \infty)}$  defined by

$$(P_t f)(x) = \mathbb{E}[f(X_t^x)], \quad t \in [0, \infty), f \in C_b^\infty(\mathbb{R}^N). \quad (1.0.10)$$

The connection presented above is very important because it shows that it is possible to get the expectation of a stochastic process by solving a partial differential equation. So, if one is interested in a numerical solution of that expectation, the Feynman-Kac Formula allows us to choose between two ways to reach it. One can either do a Monte-Carlo simulation or construct a discretisation scheme to solve the heat equation (1.0.9).

## Approximation of SDEs

There are two types of numerical methods for approximating SDEs, namely the strong and the weak approximations, (see Kloeden and Platen, 2000). The objective of the strong approximation is to produce a path-wise approximation of the solution. The weak schemes, on the other hand are appropriate when approximating the distribution of the solution at a specific instance in time. For example, in many situations, the solution  $(X_t^x)_{t \geq 0}$  of the SDE is not directly needed but the true quantity of interest is the expectation  $\mathbb{E}[f(X_T^x)]$  for some functions  $f : \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $(X_t^x)_{t \geq 0}$  being a diffusion process. Here,  $f$  might be the pay-off function of a European option. Provided that Equation (1.0.5) describes the dynamics of the underlying stock assets under a martingale measure,  $\mathbb{E}[f(X_T^x)]$  corresponds to an arbitrage-free price of the option. It is sufficient, in this case, to obtain a good approximation of the distribution of the random variable  $X^x$  rather than of its sample paths. It is mentioned in Dan and Ghazali, 2007 that Milstein was the first to show that path-wise schemes and  $\mathbb{L}^2$  estimates of the corresponding errors are not relevant in this context since the objective is to approximate the law of  $(X_t^x)_{t \geq 0}$ .

Our interest in this thesis is to study weak approximations of SDEs, that is to say the approximation of  $\mathbb{E}[f(X_T^x)]$ , for a given function  $f$  defined in  $\mathbb{R}^N$  and a fixed time  $T$ . This problem has attracted a lot of attention because of its practical importance. As already indicated above, this is equivalent in solving the heat equation numerically

$$\frac{\partial u}{\partial t}(t, x) = Lu(t, x)$$

with the initial condition  $u(0, x) = f(x)$ , where  $L$  is the partial operator defined in Equation (3.1.10) using the PDEs techniques. However, this method is efficient only when working in relatively small dimensions as opposed to the simulation approach. The simulation method is also referred to as the probabilistic method where numerical discretization schemes of some order based on the stochastic asymptotic expansion (see Section 2.2 for more details) are applied in order to construct a random variable  $X_T^{x, (n)}$  that approximates  $X_T^x$ .

**Definition 1.0.5.** *A numerical scheme is said to be of order  $m$  if there exists a positive constant  $C$  depending only on  $T, f$ , and  $x$  such that*

$$\left| \mathbb{E}(f(X_T^x)) - \mathbb{E}\left(f\left(X_T^{x, (n)}\right)\right) \right| \leq \frac{C}{n^m},$$

where the random variable  $X_T^{x, (n)}$  is obtained through the discretisation scheme  $\{X_{t_i}^{x, (n)}\}_{i=0}^n$ , with  $0 = t_0 < t_1 < \dots < t_n = T$  and  $n \in \mathbb{N}^*$ .



The most popular probabilistic method to approximate  $\mathbb{E}(f(X_T^x))$  is the Euler-Maruyama method (presented in Section 2.3) which is known to be of order at most 1 under some regularity conditions on the function  $f$ . Talay and Tubaro, 1990 showed that under the Hörmander condition (see Definition 3.1.1), this order 1 estimate holds for smooth functions  $f$ . Bally and Talay, 1996 proved this result under a much weaker hypothesis on  $f$ , namely that  $f$  needs to be only measurable and bounded, (the boundedness condition could even be relaxed). The above situation implies that if one wants to reduce the error caused by the discretisation, one has to increase the number of discretising points and one then faces the problem of numerical integration in a huge dimensional space. To overcome this problem several variance reduction techniques and Quasi-Monte Carlo methods have been proposed and we refer the interested reader to the following works (Ninomiya and Tezuka, 1996; Niederreiter, 1992). In 1999, Shigeo Kusuoka (Kusuoka, 2001) introduced a new weak approximation scheme based on Malliavin calculus and higher order stochastic Taylor expansion with discrete random variables. This method works even when the function  $f$  is Lipschitz continuous. This scheme is known as the Kusuoka approximation, it transforms the problem of calculation of  $\mathbb{E}(f(X_T^x))$  into the numerical integration over a finite set of points. Syoiti Ninomiya reported, (Ninomiya, 2003a) that for some finance problems, the Kusuoka algorithm is many thousands of times faster than the traditional Euler-Maruyama method. Lyons and Victoir extensively developed the scheme (Lyons and Victoir, 2004), by using the notion of free Lie algebra. Since then, many other schemes (see Ninomiya and Victoir, 2008; Ninomiya and Ninomiya, 2009) have been developed and are based on the Kusuoka approximation and the Cubature on Wiener space. This topic has today become an active domain of research.

## Chapter 2

# Stochastic Taylor Expansion

In this Chapter, we present the Stochastic Taylor Expansion which is the central tool in the study of numerical schemes of SDEs. After setting some notation and definitions which we routinely use throughout this thesis, we state and give a detailed proof of the stochastic Stratonovich expansion due to Platen and Wagner, 1982. This expansion provides a theoretical justification in the construction of higher order approximations schemes. The classical Euler-Maruyama scheme presented in Section 2.3 is given as an illustrative example.

### 2.1 Notations

A row vector  $\alpha = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k})$  where  $i_j \in \{0, 1, \dots, d\}$ , for  $j = 1, \dots, k$ , is called a multi-index, where  $d$  is the dimension of the Brownian motion under consideration. Notice that  $i_j$  is the  $j$ th element of the multi-index  $\alpha$ . We denote by  $\mathcal{A}$  the set of all multi-indices. Furthermore,  $\mathcal{A}_0$  and  $\mathcal{A}_1$  denote  $\mathcal{A} \setminus \{\emptyset\}$  and  $\mathcal{A} \setminus \{\emptyset, 0\}$ , respectively.

**Definition 2.1.1.** *Given a multi-index  $\alpha$ , we define*

1. *The length of the multi-index  $\alpha$  denoted by  $|\alpha|$  is equal to the number of components contained in the multi-index. That is, for  $\alpha = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k})$ ,  $|\alpha| = |(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k})| := k$ . For example,  $|(\alpha_0, \alpha_0, \alpha_0)| = 3$ ,  $|(\alpha_2, \alpha_0, \alpha_1)| = 3$ ,  $|(\alpha_0, \alpha_1, \alpha_0, \alpha_0, \alpha_2)| = 5$ .*
2. *The norm of the multi-index  $\alpha$  is the function  $\|\cdot\| : \mathcal{A} \rightarrow \mathbb{N}$  defined by*

$$\|\alpha\| = |\alpha| + \text{card}\{1 \leq j \leq |\alpha| ; i_j = 0\}.$$

*For example we have  $\|(\alpha_0, \alpha_0, \alpha_0)\| = 3 + 3 = 6$ ,  
 $\|(\alpha_2, \alpha_0, \alpha_1)\| = 3 + 1 = 4$ ,  $\|(\alpha_1, \alpha_3, \alpha_4)\| = 3$ .*

3. For two multi-indices  $\alpha = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k})$  and  $\beta = (\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_l})$ , we define the concatenation of multi-indices as the function  $*$  :  $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  given by

$$\begin{aligned}\alpha * \beta &= (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}) * (\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_l}) \\ &= (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}, \beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_l}).\end{aligned}$$

As an example, for  $\alpha = (\alpha_2, \alpha_0, \alpha_1)$ ,  $\beta = (\beta_1, \beta_3)$  we have

$$\begin{aligned}\alpha * \beta &= (\alpha_2, \alpha_0, \alpha_1, \beta_1, \beta_3) \\ \beta * \alpha &= (\beta_1, \beta_3, \alpha_2, \alpha_0, \alpha_1).\end{aligned}$$

4. The right and the left decrement are defined for  $\alpha \in \mathcal{A}$  with  $|\alpha| \geq 1$  as  $\alpha-$  and  $-\alpha$  by deleting, respectively, the last and the first component of the multi-index  $\alpha$ . This is, for  $\alpha = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k})$ ,

$$\begin{aligned}\alpha- &= (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{k-1}}) \\ -\alpha &= (\alpha_{i_2}, \alpha_{i_3}, \dots, \alpha_{i_k}).\end{aligned}$$

Thus

$$\begin{aligned}-(\alpha_0, \alpha_1, \alpha_2, \alpha_1, \alpha_3) &= (\alpha_1, \alpha_2, \alpha_1, \alpha_3) \\ (\alpha_1, \alpha_3, \alpha_0, \alpha_1, \alpha_2)- &= (\alpha_1, \alpha_3, \alpha_0, \alpha_1).\end{aligned}$$

Then  $\mathcal{A}$  becomes a semi-group with respect to the product  $(*)$  with the identity  $\emptyset$ . Also, for each  $m \geq 1$ ,  $\mathcal{A}(m) = \{\alpha \in \mathcal{A} : \|\alpha\| \leq m\}$  with  $\mathcal{A}_0(m) = \{\alpha \in \mathcal{A}_0 : \|\alpha\| \leq m\}$  and  $\mathcal{A}_1(m) = \{\alpha \in \mathcal{A}_1 : \|\alpha\| \leq m\}$ .

**Definition 2.1.2.** Let  $V$  and  $W$  be two vector fields, i.e  $V, W \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$  The Lie bracket of  $V$  and  $W$  is a new vector field denoted by  $[V, W]$  and defined by

$$[V, W] = \partial W.V - \partial V.W,$$

where  $\partial V$  is the  $N \times N$  matrix  $(\partial_j V^i)_{1 \leq i, j \leq N}$  with  $\partial_j V^i = \frac{\partial V^i}{\partial x_j}$  and  $"."$  is the matrix multiplication.

We now define the vector field concatenation  $V_{[\alpha]}$  as follows:

**Definition 2.1.3.** For all  $\alpha \in \mathcal{A}$ , we define the vector field  $V_{[\alpha]}$ , inductively by

$$\begin{aligned}V_{[\emptyset]} &= 0, \quad V_{[(\alpha_i)]} = V_i, \quad i = 0, 1, \dots, d \\ V_{[\alpha]} &= [V_{[\alpha-]}, V_{[i_k]}] \quad \text{for } \alpha = (\alpha_{i_1}, \dots, \alpha_{i_k})\end{aligned}$$

where  $[V_{[\alpha-]}, V_{[i_k]}]$  is the Lie bracket of  $V_{[\alpha-]}$  and  $V_{[\alpha_{i_k}]}$ .

## 2.2 Iterated Stratonovich Integrals

The iterated Itô-Stratonovich integrals of a  $d$ -dimensional Brownian motion are important given the fact that they appear in the stochastic Taylor expansion. They play a similar role as polynomials do in the deterministic Taylor expansion. For this reason, they have a central role in the numerical analysis of SDEs, in particular for the construction of higher order (order  $\geq 2$ ) approximation schemes. Moreover, the recent theory of Terry Lyons, "rough path theory" (Lyons, 1998) has shown the significant position of the iterated integrals in the theory of stochastic differential equations.

We recall that given a stochastic process  $(X_t^x)_{t \in [0, T]}$  satisfying Equation (1.0.1), and a smooth function  $f$ ,

$$f(X_t^x) = f(x) + \int_0^t \sum_{i=0}^d V_i f(X_s^x) \circ dB^i(s), \quad (2.2.1)$$

where  $X_0^x = x$  is the initial value of  $X$ . As usual,  $V_0, \dots, V_d \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$  are  $C^\infty$ -bounded vector fields on  $\mathbb{R}^N$ . Below is Itô's idea to define Stratonovich integrals within the class of Itô processes (Kuo, 2005).

**Definition 2.2.1.** *Let  $(X_t)_{t \in [0, T]}$  and  $(Y_t)_{t \in [0, T]}$  be two Itô processes. We define the Stratonovich integral of  $X_t$  with respect to  $Y_t$  as*

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2} \langle X, Y \rangle_t \quad \text{for } 0 \leq t \leq T$$

where  $\int_0^t X_s dY_s$  is the Itô integral and  $\langle X, Y \rangle$  is the quadratic covariation process.

As an example of the above definition, we have that

$$\int_0^t X_s \circ dX_s = \frac{1}{2} X_t^2$$

provided that  $X_0 = 0$ , which shows that the Stratonovich integrals behave like the integral in classical calculus. The Stratonovich integral does not have the martingale property which turns out to be crucial in stochastic analysis, but it obeys the usual transformation rules of classical calculus and this is the reason for our interest in the integral in the Stratonovich form.

The following notation will help us to write down many formulas in a more concise way.

**Definition 2.2.2.** *Let  $f : [0, T] \rightarrow \mathbb{R}^d$ ,  $f(t) = (f^1(t), \dots, f^d(t))$ . We define the  $0^{\text{th}}$  component of  $f$  by setting*

$$f^0(t) = t, \quad t \in [0, T].$$

$f$  is said to have bounded variation components if for  $i = 1, \dots, d$ ,  $f^i : [0, T] \rightarrow \mathbb{R}$  is of bounded variation, that is to say for  $i = 0, 1, \dots, d$ ,

$$\sup_{\pi} \sum_{j=1}^m |f^i(t_j) - f^i(t_{j-1})| < \infty$$

where each  $\pi = \{t_0, \dots, t_m\}$ ,  $0 = t_0 < t_1 < \dots < t_j < t_{j+1} < \dots < t_m = T$  is a partition of  $[0, T]$ . In particular, for a  $d$ -dimensional Brownian motion, we set  $B^0(t) = t$ .

We define the iterated Stratonovich integrals in the following way

**Definition 2.2.3.** Let  $t \in [0, T]$ ,  $\alpha \in \mathcal{A}$ .  $B^{\circ\alpha}(t)$  is defined inductively by

$$B^{\circ\emptyset}(t) = 1, \quad B^{\circ(\alpha_i)}(t) = B^i(t), \quad i \in \{0, \dots, d\}$$

for  $\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k})$

$$\begin{aligned} B^{\circ\alpha}(t) &= \int_0^t B^{\circ\alpha-}(s) \circ dB^{i_k}(s) \\ &= \int_{0 < t_1 < \dots < t_k < t} \circ dB_{t_1}^{i_1} \circ \dots \circ dB_{t_k}^{i_k} \\ &= \int_0^t \int_0^{t_k} \dots \int_0^{t_2} \circ dB_{t_1}^{i_1} \circ \dots \circ dB_{t_k}^{i_k} \end{aligned}$$

and for a given smooth function  $f$  and an integrable process  $(X_t^x)_{t \in [0, T]}$ ,

$$B^{\circ\alpha}(f(X_t^x)) = \int_{0 < t_1 < \dots < t_k < t} f(X_{t_1}^x) \circ dB_{t_1}^{i_1} \circ \dots \circ dB_{t_k}^{i_k}.$$

More generally, we make the following definition:

**Definition 2.2.4.** Let  $f : [0, T] \rightarrow \mathbb{R}^d$  be either a deterministic function with bounded variation components, or a  $d$ -dimensional standard Brownian motion. For  $\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}_0$ , we define the iterated integral of  $f$  by

$$\begin{aligned} f^{\circ\alpha}(t) &= f^{(\alpha_{i_1}, \dots, \alpha_{i_k})}(t) \\ &= \int_{0 < t_1 < \dots < t_k < t} df^{i_1}(t_1) \dots df^{i_k}(t_k) \\ &= \int_0^t \int_0^{t_k} \dots \int_0^{t_2} df^{i_1}(t_1) \dots df^{i_k}(t_k). \end{aligned}$$

Moreover,  $f^{\circ\emptyset}(t) = 1$ . If  $f$  is a Brownian motion, the integrals are understood in the sense of iterated Stratonovich integrals.

In the same way we define the Itô iterated integrals as

$$\begin{aligned} D^\alpha(t) &= \int_{0 < t_1 < \dots < t_k < t} dB_{t_1}^{i_1} \dots dB_{t_k}^{i_k} \\ D^\alpha(f(X_t^x)) &= \int_{0 < t_1 < \dots < t_k < t} f(X_{t_1}^x) dB_{t_1}^{i_1} \dots dB_{t_k}^{i_k}. \end{aligned}$$

Notice that  $B^{\circ\alpha}(t)$  is equal in law to

$$t^{\|\alpha\|/2} B^{\circ\alpha}(1) = t^{\|\alpha\|/2} \int_{0 < t_1 < \dots < t_k < 1} \circ dB_{t_1}^{i_1} \circ \dots \circ dB_{t_k}^{i_k}$$

which is a generalization of the fact that  $B_t$  is equal in law to  $\sqrt{t}B_1$  for Brownian motion. To derive a general relation between iterated Stratonovich integrals and Itô integrals, we need the following relation between multi-indices:

**Definition 2.2.5.** (Lyons and Gyurko, July 23, 2008) Let  $\alpha \in \mathcal{A}$  be a multi-index, let  $(\alpha^1, \dots, \alpha^k) \in \mathcal{A}^k$  be a partition of  $\alpha$  i.e  $\alpha = \alpha^1 * \alpha^2 * \dots * \alpha^k$  for  $k \in \mathbb{N}$ , such that  $\|\alpha^i\| \leq 2$  for  $i = 1, \dots, k$ . Suppose that there exists a multi-index  $\beta$  which can be partitioned such that  $\beta = \beta^1 * \dots * \beta^k$ . We will say that  $\beta$  is related to  $\alpha$  through the partitions  $(\beta^1, \dots, \beta^k)$  and  $(\alpha^1, \dots, \alpha^k)$  of length  $k$  if for each  $i = 1, \dots, k$ ,

$$\alpha^i = \beta^i \quad \text{with} \quad \|\alpha^i\| = 1 = \|\beta^i\|$$

or

$$\alpha^i = (\alpha_l^i, \alpha_l^i) \quad \text{and} \quad \beta^i = (\beta_0^i) \quad \text{for some} \quad l \in \{1, \dots, d\}.$$

We will denote this relationship by  $\alpha \sim_k \beta$ , where  $k$  is the number of sub-indices in the related partitions.

If  $\alpha \sim_k \beta$ , we define the function  $\nu : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{N}$  by

$$\nu(\alpha, \beta) := \text{card} \{i | 1 \leq i \leq k, \alpha^i \neq \beta^i\}.$$

From Definition 2.2.1, we have the following useful equation:

$$\int_0^t f(X_s^x) \circ dB_s^i = \int_0^t f(X_s^x) dB_s^i + (1 - \delta_{i,0}) \frac{1}{2} \int_0^t V_i f(X_s^x) ds$$

valid for a smooth function  $f$  and an integrable process  $(X_t^x)_{t \in [0, T]}$ .  $\delta_{i,0} = 1$  if  $i = 0$  and  $\delta_{i,0} = 0$  if  $i \neq 0$ . By iterating this equation, we obtain the following Lemma:

**Lemma 2.2.6.** (Lyons and Gyurko, July 23, 2008) For any multi-index  $\alpha$ ,

$$\begin{aligned} B^{\circ\alpha}(f(X_t^x)) &= \sum_{\substack{\alpha \sim_k \beta \\ k \in \mathbb{N}}} \frac{1}{2^{\nu(\alpha, \beta)}} D^\beta(f(X_t^x)) \\ &\quad + (1 - \delta_{i_1, 0}) \frac{1}{2} \sum_{\substack{-\alpha \sim_k \beta \\ k \in \mathbb{N}}} D^\beta(D^0(V_{\alpha_1} f(X_t^x))) \end{aligned}$$

where  $D^\beta(f(X_t^x)) = \int_{0 < t_1 < \dots < t_k < t} f(X_{t_1}^x) dB_{t_1}^{j_1} \dots dB_{t_k}^{j_k}$  is the Itô iterated integral,  $\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k})$  and  $\beta = (\beta_{j_1}, \dots, \beta_{j_k})$ .

**Remark 2.2.7.** Lemma 2.2.6 expresses the fact that the iterated integral

$$\int_{0 < t_0 < t_1 < \dots < t_k < t} V_{i_0} \dots V_{i_k} f(X_{t_0}^x) \circ dB_{t_0}^{i_0} \circ \dots \circ dB_{t_k}^{i_k}$$

is equal to a sum of terms in the form

$$\int_{0 < t_0 < t_1 < \dots < t_k < t} V_{i_0} \dots V_{i_k} f(X_{t_0}^x) dB_{t_0}^{i_0} \dots dB_{t_k}^{i_k}$$

and terms in the form

$$\frac{1}{2} \int_{0 < t_0 < t_1 < \dots < t_k < t} V_{i_j} V_{i_0} \dots V_{i_k} f(X_{t_0}^x) dB_{t_0}^{i_0} \dots dB_{t_{j-1}}^{i_{j-1}} dt_j dB_{t_{j+1}}^{i_{j+1}} \dots dB_{t_k}^{i_k}.$$

We introduce this useful Lemma which gives using the Itô isometry and the Hölder inequality, the  $\mathbb{L}^2$  boundedness of the Itô iterated integrals. It is a key point in the proof of Proposition 2.2.9.

**Lemma 2.2.8.** Given a multi-index  $\alpha = (\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}$  and a bounded smooth function  $f$ , we have

$$\mathbb{E} \left[ \left( \int_{0 < t_0 < t_1 < \dots < t_k < t} V_{i_0} \dots V_{i_k} f(X_{t_0}^x) dB_{t_0}^{i_0} \dots dB_{t_k}^{i_k} \right)^2 \right] \leq t^{\|\alpha\|} \left\| V_{i_0} \dots V_{i_k} f \right\|_\infty^2 \quad (2.2.2)$$

*Proof.* We prove this lemma by induction on  $k$ .

For  $k = 0$ , we distinguish between two cases

1.  $i_0 = i_k = 0$

Let  $h$  be a bounded smooth function

$$\begin{aligned} \mathbb{E} \left[ \left( \int_0^t h(X_s^x) dB_s^0 \right)^2 \right] &= \mathbb{E} \left[ \left( \int_0^t h(X_s^x) ds \right)^2 \right] \\ &\leq \mathbb{E} \left[ \left( \int_0^t |1| \cdot |h(X_s^x)| ds \right)^2 \right] \end{aligned}$$

which by Hölder inequality is

$$\begin{aligned} &\leq \mathbb{E} \left[ \left( \sqrt{\int_0^t 1^2 ds} \cdot \sqrt{\int_0^t h(X_s^x)^2 ds} \right)^2 \right] \\ &= t \int_0^t \mathbb{E} [h(X_s^x)^2] ds \\ &\leq t \int_0^t \|h\|_\infty^2 ds \\ &= t^2 \|h\|_\infty^2. \end{aligned}$$

2.  $i_0 = i_k = i$ ,  $i \in \{1, \dots, d\}$

By Itô isometry, we obtain

$$\begin{aligned} \mathbb{E} \left[ \left( \int_0^t h(X_s^x) dB_s^i \right)^2 \right] &= \mathbb{E} \left[ \int_0^t h(X_s^x)^2 ds \right] \\ &= \int_0^t \mathbb{E} \left( h(X_s^x)^2 \right) ds \\ &\leq t \|h\|_\infty^2. \end{aligned}$$

Setting  $h = V_{i_0} f(X_{t_0}^x)$ , we have the proof for  $k = 0$ . We now proceed to the induction step, we suppose that we have the result for all  $p \leq k$  for a fixed  $k \in \mathbb{N}$ . We want to prove that

$$\begin{aligned} \mathbb{E} \left[ \left( \int_{0 < t_0 < \dots < t_k < t_{k+1} < t} V_{i_0} \dots V_{i_k} V_{i_{k+1}} f(X_{t_0}^x) dB_{t_0}^{i_0} \dots dB_{t_k}^{i_k} dB_{t_{k+1}}^{i_{k+1}} \right)^2 \right] \\ \leq t^{\|\alpha * \alpha_{k+1}\|} \|V_{i_0} \dots V_{i_k} f\|_\infty^2 \end{aligned}$$

Considering the same development as before, and replacing  $h$  by  $\int_{0 < t_0 < \dots < t_k < t_{k+1}} V_{i_0} \dots V_{i_k} V_{i_{k+1}} f(X_{t_0}^x) dB_{t_0}^{i_0} \dots dB_{t_k}^{i_k}$ , we obtain the result.  $\square$

The stochastic Taylor expansion generalizes both the deterministic Taylor formula and the Itô formula. It was first introduced by Platen and Wagner, 1982 for the class of Itô processes and based on the use of multiple stochastic integrals. It is the starting point of stochastic numerical analysis. There are several possibilities for such an expansion. One is based on the iterated application of the Itô formula and is called the Itô-Taylor expansion. The other one takes the Stratonovich representation of the process into consideration and is called the Stratonovich-Taylor expansion, see Kloeden and Platen, 2000 for details on stochastic Taylor expansions and many methods originating from it. The Stratonovich-Taylor expansion has a simpler structure which makes it a more natural generalization of the deterministic Taylor formula and more convenient to use in stochastic numerical analysis. The following proposition states the stochastic Stratonovich-Taylor expansion:

**Proposition 2.2.9.** *Let  $m$  be a natural number and  $f \in C_b^{m+1}(\mathbb{R}^N)$ . Consider the solution  $X_t^x$  of the SDE (1.0.1). Then*

$$f(X_t^x) = f(x) + \sum_{\alpha \in \mathcal{A}(m)} V_{i_1} \dots V_{i_k} f(x) \int_{0 < t_1 < \dots < t_k < t} \circ dB_{t_1}^{i_1} \circ \dots \circ dB_{t_k}^{i_k} + R_m(t, x, f) \quad (2.2.3)$$

where the remainder term  $R_m$  satisfies

$$\sup_{x \in \mathbb{R}^N} \sqrt{\mathbb{E} \left( R_m(t, x, f)^2 \right)} \leq C t^{\frac{m+1}{2}} \sup_{\alpha \in \mathcal{A}(m+2) \setminus \mathcal{A}(m)} \|V_{i_1} \dots V_{i_k} f\|_\infty, \quad (2.2.4)$$



for some positive constant  $C$  depending only on  $d$  and  $m$ , provided that  $t \leq 1$ .

**Remark 2.2.10.** The previous inequality means that the remainder is of order  $O(t^{m+1/2})$ .

*Proof.* The key idea of the proof is the iterated application of the Itô formula. Indeed, for a smooth function  $f$  and a process  $(X_t^x)_{t \in [0, T]}$ , the Stratonovich form of this formula yields

$$f(X_t^x) = f(x) + \sum_{i=0}^d \int_0^t V_i f(X_s^x) \circ dB_s^i. \quad (2.2.5)$$

Since each function  $V_i f : \mathbb{R}^N \rightarrow \mathbb{R}$  is smooth, using equation (2.2.5) we obtain,

$$V_i f(X_s^x) = V_i f(x) + \sum_{j=0}^d \int_0^s V_j V_i f(X_u^x) \circ dB_u^j. \quad (2.2.6)$$

We first establish that the remainder term  $R_m(t, x, f)$  can be written as

$$R_m(t, x, f) = \sum_{\substack{\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m) \\ (\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_k}) \notin \mathcal{A}(m)}} \int_{0 < t_0 < t_1 < \dots < t_k < t} V_{i_0} \cdots V_{i_k} f(X_{t_0}^x) \circ dB_{t_0}^{i_0} \circ \dots \circ dB_{t_k}^{i_k}. \quad (2.2.7)$$

This fact is proved by induction on  $m$ . Indeed, for  $m = 1$ , taking in account Equations (2.2.5) and (2.2.6), we obtain

$$f(X_t^x) = f(x) + \sum_{i=0}^d V_i f(x) \int_0^t \circ dB_s^i + \sum_{i=0}^d \sum_{j=0}^d \int_0^t \int_0^s V_j V_i f(X_u^x) \circ dB_u^j \circ dB_s^i$$

that is

$$\begin{aligned} f(X_t^x) &= f(x) + \sum_{i=1}^d V_i f(x) \int_0^t \circ dB_s^i + \int_0^t V_0 f(x) ds \\ &\quad + \sum_{i=0}^d \sum_{j=0}^d \int_0^t \int_0^s V_j V_i f(X_u^x) \circ dB_u^j \circ dB_s^i \\ &= f(x) + \sum_{\alpha = (\alpha_i) \in \mathcal{A}(1)} V_i f(x) \\ &\quad + \sum_{\substack{\alpha = (\alpha_i) \in \mathcal{A}(1) \\ (\alpha_j, \alpha_i) \notin \mathcal{A}(1)}} \int_0^t \int_0^s V_j V_i f(X_u^x) \circ dB_u^j \circ dB_s^i. \end{aligned}$$

We now proceed to the induction step. We assume that the formula is true for all  $p \leq m$  for a fixed  $m \in \mathbb{N}$ , this means,

$$f(X_t^x) = T_m(t, x, f) + R_m(t, x, f),$$

where

$$T_m(t, x, f) = f(x) + \sum_{\alpha \in \mathcal{A}(m)} V_{i_1} \cdots V_{i_k} f(x) \int_{0 < t_1 < \cdots < t_k < t} \circ dB_{t_1}^{i_1} \circ \cdots \circ dB_{t_k}^{i_k}$$

and  $R_m(t, x, f)$  is as in Equation (2.2.7). Let us prove the formula for  $m+1$ . By replacing  $f$  by  $V_{i_0} \cdots V_{i_k} f(X_{t_0}^x)$  in Equation (2.2.5), we get

$$V_{i_0} \cdots V_{i_k} f(X_{t_0}^x) = V_{i_0} \cdots V_{i_k} f(x) + \sum_{i=0}^d \int_0^{t_0} V_i V_{i_0} \cdots V_{i_k} f(X_s^x) \circ dB_s^i \quad (2.2.8)$$

and by applying this identity to the induction hypothesis  $R_m(t, x, f)$ , we obtain

$$\begin{aligned} f(X_t^x) &= T_m(t, x, f) \\ &+ \sum_{\substack{\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m) \\ (\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_k}) \notin \mathcal{A}(m)}} V_{i_0} \cdots V_{i_k} f(x) \int_{0 < t_0 < t_1 < \cdots < t_k < t} \circ dB_{t_0}^{i_0} \circ \cdots \circ dB_{t_k}^{i_k} \\ &+ \sum_{\substack{\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m) \\ (\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_k}) \notin \mathcal{A}(m)}} \sum_{i=0}^d \int_{0 < t_0 < t_1 < \cdots < t_k < t} \int_0^{t_0} V_i V_{i_0} \cdots V_{i_k} f(X_s^x) \\ &\quad \circ dB_s^i \circ dB_{t_0}^{i_0} \circ \cdots \circ dB_{t_k}^{i_k}. \end{aligned}$$

$\alpha \in \mathcal{A}(m)$  and  $(\alpha_{i_0}, \alpha) \notin \mathcal{A}(m)$  means that  $\|\alpha\| \leq m$  and  $\|(\alpha_{i_0}, \alpha)\| > m$ ; There are three possibilities:

1.  $\|\alpha\| = m$  and  $i_0 \in \{1, \dots, d\}$
2.  $\|\alpha\| = m-1$  and  $i_0 = 0$
3.  $\|\alpha\| = m$  and  $i_0 = 0$ .

For the first and the second cases, we have that  $(\alpha_{i_0}, \alpha) \in \mathcal{A}(m+1)$ , thus,

$$V_{i_0} \cdots V_{i_k} f(x) \int_{0 < t_0 < t_1 < \cdots < t_k < t} \circ dB_{t_0}^{i_0} \circ \cdots \circ dB_{t_k}^{i_k}$$

occurs in  $T_{m+1}(t, x, f)$  and

$$\int_{0 < s < t_0 < t_1 < \cdots < t_k < t} V_i V_{i_0} \cdots V_{i_k} f(X_s^x) \circ dB_s^i \circ dB_{t_0}^{i_0} \circ \cdots \circ dB_{t_k}^{i_k}$$

occurs in  $R_{m+1}(t, x, f)$ , for any  $i \in \{0, 1, \dots, d\}$ . In the last case,  $(i_0, \alpha) \in \mathcal{A}(m+2) \setminus \mathcal{A}(m+1)$ , and the corresponding term is

$$\begin{aligned} & V_0 V_{i_1} \cdots V_{i_k} f(x) \int_{0 < t_0 < t_1 < \cdots < t_k < t} \circ dB_{t_0}^0 \circ \cdots \circ dB_{t_k}^{i_k} \\ & + \sum_{i=0}^d \int_{0 < s < t_0 < t_1 < \cdots < t_k < t} V_i V_0 V_{i_1} \cdots V_{i_k} f(X_s^x) \circ dB_s^i \circ dt_0 \circ \cdots \circ dB_{t_k}^{i_k} \\ = & \int_{0 < t_0 < t_1 < \cdots < t_k < t} \left[ V_0 V_{i_1} \cdots V_{i_k} f(x) + \sum_{i=0}^d \int_0^{t_0} V_i V_0 V_{i_1} \cdots V_{i_k} f(X_s^x) \circ dB_s^i \right] \\ & \circ dt_0 \circ \cdots \circ dB_{t_k}^{i_k} \end{aligned}$$

which is, regarding Equation (2.2.8) equal to

$$\int_{0 < t_0 < t_1 < \cdots < t_k < t} V_0 V_{i_1} \cdots V_{i_k} f(X_{t_0}^x) \circ dt_0 \circ \cdots \circ dB_{t_k}^{i_k}$$

and this term occurs in  $R_{m+1}(t, x, f)$ . And so, we have the desired result.

The proof of Inequality (2.2.4) is based on Remark 2.2.7 and Lemma 2.2.8. For the terms in the form

$$\frac{1}{2} \int_{0 < t_0 < \cdots < t_k < t} V_{i_j} V_{i_0} \cdots V_{i_k} f(X_{t_0}^x) dB_{t_0}^{i_0} \cdots dB_{t_{j-1}}^{i_{j-1}} dt_j dB_{t_{j+1}}^{i_{j+1}} \cdots dB_{t_k}^{i_k},$$

we show, using the same idea as in the Proof of Lemma 2.2.8 that,

$$\begin{aligned} \mathbb{E} \left[ \left( \int_{0 < t_0 < t_1 < \cdots < t_k < t} V_{i_j} V_{i_0} \cdots V_{i_k} f(X_{t_0}^x) dB_{t_0}^{i_0} \cdots dB_{t_{j-1}}^{i_{j-1}} dt_j dB_{t_{j+1}}^{i_{j+1}} \cdots dB_{t_k}^{i_k} \right)^2 \right] \\ \leq t^{\|\alpha\|+1} \|V_{i_j} V_{i_0} \cdots V_{i_k} f\|_{\infty}^2 \end{aligned} \quad (2.2.9)$$

Let us recall our original problem which is to find the upper-bound of

$$\begin{aligned} R_m(t, x, f) = & \sum_{\substack{\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m) \\ (\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_k}) \notin \mathcal{A}(m)}} \int_{0 < t_0 < t_1 < \cdots < t_k < t} V_{i_0} \cdots V_{i_k} f(X_{t_0}^x) \\ & \circ dB_{t_0}^{i_0} \circ \cdots \circ dB_{t_k}^{i_k}. \end{aligned}$$

Taking Equations (2.2.2) and (2.2.9) into account, we have

$$\begin{aligned} \mathbb{E} \left[ (R_m(t, x, f))^2 \right] \leq & c_1 t^{m+1} \sum_{(\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m+1) \setminus \mathcal{A}(m)} \|V_{i_0} \cdots V_{i_k} f\|_{\infty}^2 \\ & + c_2 t^{m+2} \sum_{(\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m+2) \setminus \mathcal{A}(m+1)} \|V_{i_0} \cdots V_{i_k} f\|_{\infty}^2 \end{aligned}$$

that is,

$$\begin{aligned} \left( \mathbb{E} \left[ (R_m(t, x, f))^2 \right] \right)^{\frac{1}{2}} &\leq c_1 t^{\frac{m+1}{2}} \sum_{(\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m+1) \setminus \mathcal{A}(m)} \|V_{i_1} \cdots V_{i_k} f\|_{\infty} \\ &\quad + c_2 t^{\frac{m+2}{2}} \sum_{(\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m+2) \setminus \mathcal{A}(m+1)} \|V_{i_1} \cdots V_{i_k} f\|_{\infty}. \end{aligned}$$

Considering the fact that for  $t \leq 1$ ,  $t^{\frac{m+2}{2}} \leq t^{\frac{m+1}{2}}$ . This implies that

$$\begin{aligned} \left( \mathbb{E} \left[ (R_m(t, x, f))^2 \right] \right)^{\frac{1}{2}} &\leq c_1 t^{\frac{m+1}{2}} \sum_{(\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m+1) \setminus \mathcal{A}(m)} \|V_{i_1} \cdots V_{i_k} f\|_{\infty} \\ &\quad + c_2 t^{\frac{m+1}{2}} \sum_{(\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m+2) \setminus \mathcal{A}(m+1)} \|V_{i_1} \cdots V_{i_k} f\|_{\infty} \\ &\leq C t^{\frac{m+1}{2}} \sup_{(\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m+2) \setminus \mathcal{A}(m)} \|V_{i_1} \cdots V_{i_k} f\|_{\infty}. \end{aligned}$$

We then conclude that

$$\sup_{x \in \mathbb{R}^N} \sqrt{\mathbb{E} \left[ (R_m(t, x, f))^2 \right]} \leq C t^{\frac{m+1}{2}} \sup_{(\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m+2) \setminus \mathcal{A}(m)} \|V_{i_1} \cdots V_{i_k} f\|_{\infty}.$$

□

**Remark 2.2.11.**

- The Taylor Stratonovich expansion remains correct if we replace  $\circ dB_t$  by any continuous semi-martingales.
- Furthermore, the expansion still works when  $\circ dB_t$  is replaced by  $dx_t$  where  $x_t$  is a rough path (see Lyons (1998) for more details).

Notice that these expansions once again underline the importance of iterated integrals.

## 2.3 Euler-Maruyama Scheme

The aim of this section is to briefly introduce the Euler-Maruyama scheme which can be directly deduced from the stochastic Taylor expansion. It is the most familiar and simple numerical approximation scheme. The standard reference for the approximation of SDEs is Kloeden and Platen, 2000. Let us once again consider the SDE (1.0.1),

$$dX_t^x = \bar{V}_0(X_t^x) dt + \sum_{j=1}^d V_j(X_t^x) \cdot dB_t^j. \quad (2.3.1)$$

The stochastic Euler method for solving this equation is given as follows: We first fix a partition  $0 = s_0 < s_1 < \dots < s_N = T$  of  $[0, T]$ , with size  $N$ . Then we define recursively a discrete-time stochastic process  $(X_k^N)_{k=0}^N$  as follows:

$$\begin{aligned} X_0^N &= x \\ X_{k+1}^N &= X_k^N + \bar{V}_0(X_k^N) \Delta s_k + \sum_{i=1}^d V_i(X_k^N) \Delta B_k^i \end{aligned}$$

$k = 0, 1, \dots, N-1$ . Here,  $\Delta s_k = s_{k+1} - s_k = T/N$ , for  $k = 0, 1, \dots, N-1$  is called the step size, and  $\Delta B_k^i = B_{s_{k+1}}^i - B_{s_k}^i$ , for  $k = 0, 1, \dots, N-1$ . Notice that  $\Delta B_k^i$  is equal in law to  $\sqrt{\Delta s_k} Y$ , where  $Y$  is a one-dimensional standard normal random variable. One can then show that for an arbitrary  $\mathbf{C}^4$  function  $f$ ,

$$|\mathbb{E}(f(X_N^N)) - \mathbb{E}(f(X_T^x))| = \mathcal{O}\left(\frac{1}{N}\right),$$

that is to say, the Euler-Maruyama scheme is of order 1. A way to obtain higher order approximation schemes (order greater than 1) is based on taking into account more terms in the stochastic Taylor expansion. In the general case, one needs to understand how to weakly approximate the increments of the Brownian motion together with its iterated integrals. Considering the stochastic Taylor expansion in terms of the Stratonovich integrals presented in Proposition 2.2.9, one could think of the extension of the Euler method and obtain

$$X_{k+1}^N = X_k^N + \sum_{\alpha \in \mathcal{A}(m)} (V_{i_1} \dots V_{i_k})(X_k^N) \times \int_{s_k}^{s_{k+1}} \int_{s_k}^{t_k} \dots \int_{s_k}^{t_2} \circ dB_{t_1}^{i_1} \dots \circ dB_{t_k}^{i_k}.$$

However, the joint density of the iterated stochastic integral is not known, so this make the implementation of this method difficult. High order approximation based on the stochastic Taylor expansion was successfully done by Talay (1990), and recently by Kusuoka (2001), Kusuoka and Ninomiya, 2004, and then generalised with the method Cubature on Wiener space by Lyons and Victoir, 2004. This will be detailed in the next chapters.

**Remark 2.3.1.** Consider a scheme  $(X_k^N)_{k=0}^N$  of order  $p$  with size  $N$  such that for any smooth function  $f$ , there exists a constant  $K_f$  such that

$$\mathbb{E}[f(X_N^N)] = \mathbb{E}[f(X_T^x)] + K_f \frac{1}{N^p} + \mathcal{O}\left(\frac{1}{N^{p+1}}\right).$$

Then, we can obtain a scheme of order  $p+1$  by the method of Romberg Extrapolation as follows: We consider our approximation scheme of order  $p$  with size

$N$  and  $2N$ , that is we obtain two processes  $(X_k^N)_{k=0}^N$  and  $(X_k^{2N})_{k=0}^{2N}$ . It can be shown that

$$\frac{2^p}{2^p - 1} \mathbb{E} [f(X_{2N}^{2N})] - \frac{1}{2^p - 1} \mathbb{E} [f(X_N^N)] \quad (2.3.2)$$

provides a scheme of order  $p + 1$ .

Talay and Tubaro, 1990 have shown that the Romberg Extrapolation method can be applied to the Euler-Maruyama scheme.

## Chapter 3

# Kusuoka scheme

Options pricing problems in mathematical finance are related to the numerical computation of expectations of diffusion processes. For European options, one needs to compute  $\mathbb{E}[f(X_T^x)]$  where  $f$  is a  $\mathbb{R}$ -valued function defined on  $\mathbb{R}^N$  and  $X_T^x$  is the value at expiration time  $T \in ]0, \infty[$  of a diffusion process  $(X_t^x)_{0 \leq t \leq T}$  given by the following stochastic differential equation written in the Stratonovich form:

$$X_t^x = x + \sum_{j=0}^d \int_0^t V_j(X_s^x) \circ dB^j(s) \quad (3.0.1)$$

where  $V_j \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ ,  $B^0(t) = t$  and  $(B^1(t), \dots, B^d(t))$  is the

$d$ -dimensional standard Brownian motion. It is well known that under the Hörmander condition on a diffusion process, the Euler-Maruyama method gives a good approximation even for a bounded measurable function  $f$ . It is also known that in this case the accuracy that one obtains is proportional to the width of a discretisation unit. In other words, if one wants to make the error due to the discretisation smaller, one has to increase the number of discretising points and then, we find that we face the problem of numerical integration in a huge dimensional space.

In 2001, Kusuoka introduced a new simulation scheme based on Malliavin Calculus, higher-order stochastic Taylor expansion based on Lie algebra and involves a non-uniform discretization of the time interval. This method is constructed by assuming the so-called UFG condition (see definition 3.1.1) which is weaker than the Hörmander condition.

In this chapter, we present the main idea of the Kusuoka approximation scheme taken from Kusuoka (2001), Kusuoka (2004) and Kusuoka and Nishimura, 2004. It is based on the introduction of a " $m$ -moment similar" family of random variables with the property that the expectations of these random variables are the same as the Stratonovich iterated integrals of Brownian motion of degree less than or equal to  $m$  (see Definition 3.1.4). From this family

of random variables, a Markov operator which approximates  $P_T$  (1.0.10) is constructed (see Definition 3.1.10). In the last section, we implement the Kusuoka approximation to the pricing of an Asian call option. Moreover, we compare the numerical results obtained using the Kusuoka scheme with the traditional Euler-Maruyama method. Our results show that, using the Kusuoka scheme, we can reduce the number of dimensions required for the simulation and also achieve faster calculations and this agrees with Ninomiya (2003a) and Ninomiya (2003b).

### 3.1 Kusuoka's Approximation Scheme

Considering the notation defined in Sections 2.1 and 2.2, we make the following definitions:

**Definition 3.1.1.**

1. Let  $\alpha \in \mathcal{A}_1$ , the vector field  $V_{[\alpha]}$  is said to satisfy the UFG condition if there exist some functions  $\varphi_{\alpha,\beta} \in C_b^\infty(\mathbb{R}^N)$ , for  $\ell \in \mathbb{N}$  and  $\beta \in \mathcal{A}_1(\ell)$  such that

$$V_{[\alpha]} = \sum_{\beta \in \mathcal{A}_1(\ell)} \varphi_{\alpha,\beta} V_{[\beta]}. \quad (3.1.1)$$

2. Consider a vector field  $V$ . The Uniform Hörmander condition (UH) is said to be satisfied if there is an integer  $\ell$  and a constant  $c > 0$  such that

$$\sum_{\beta \in \mathcal{A}_1(\ell)} \langle V_{[\beta]}, \varepsilon \rangle^2 \geq c \|\varepsilon\|^2$$

for all  $x, \varepsilon \in \mathbb{R}^N$ , where  $\langle V, \varepsilon \rangle = \sum_{i=1}^N V^i \cdot \varepsilon^i$  for  $V \in C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$ .

**Remark 3.1.2.** Let us think of  $C_b^\infty(\mathbb{R}^N)$ -module  $\mathbf{M} = \sum_{\alpha \in \mathcal{A}_0} C_b^\infty(\mathbb{R}^N) V_{[\alpha]}$ . Then the UFG condition is equivalent to the assumption that  $\mathbf{M}$  is finitely generated as a  $C_b^\infty(\mathbb{R}^N)$ -module.

**Lemma 3.1.3.** (Dan and Ghazali, 2007) The UH condition implies the UFG condition.

*Proof.* We follow the proof of Proposition 5.1 in (Ter Elst and Robinson, 2009). Suppose that the Uniform Hörmander condition is satisfied, that is, there exists  $\ell \in \mathbb{N}$  and  $c > 0$  such that

$$\sum_{\beta \in \mathcal{A}_1(\ell)} \langle V_{[\beta]}, \varepsilon \rangle^2 \geq c \|\varepsilon\|^2.$$

Let  $\alpha \in \mathcal{A}_1$ , then we are looking for  $\varphi_{\alpha,\beta} \in C_b^\infty(\mathbb{R}^N)$  such that

$$V_{[\alpha]} = \sum_{\beta \in \mathcal{A}_1(\ell)} \varphi_{\alpha,\beta} V_{[\beta]}.$$



$V_{[\alpha]}$  is a  $C_b^\infty$ -vector field so it can be written as

$$V_{[\alpha]} = \sum_{i=1}^N V_{[\alpha]}^i \frac{\partial}{\partial x_i},$$

that is

$$V_{[\alpha]} = a \cdot \partial, \quad (3.1.2)$$

where  $a = (V_{[\alpha]}^1, \dots, V_{[\alpha]}^N)$  and  $\partial = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N})'$ . Also, for all  $\beta \in \mathcal{A}_1(\ell)$ , we have that  $V_{[\beta]} = a_\beta \cdot \partial$ , where  $a_\beta = (V_{[\beta]}^1, \dots, V_{[\beta]}^N)$ . We set  $t = \text{card}(\mathcal{A}_1(\ell))$ , we obtain

$$X = B \cdot \partial.$$

Here,  $X = (V_{[\beta_1]}, \dots, V_{[\beta_t]})'$  and  $B$  is the matrix  $(a_{\beta_1}, \dots, a_{\beta_t})'$ , and each  $\beta_i \in \mathcal{A}_1(\ell)$  for  $i = 1, \dots, t$ . By assumption, for all  $\varepsilon \in \mathbb{R}^N$ ,  $\varepsilon^T B^T B \varepsilon > c|\varepsilon|^2$ , this implies that  $B^T B$  has a positive determinant and so is invertible. Let  $A = B^T B$ , we have

$$B^T X = A \partial,$$

that is

$$\partial = A^{-1} B^T X.$$

By replacing it in equation (3.1.2), we obtain

$$V_{[\alpha]} = a \cdot \partial \quad (3.1.3)$$

$$= a \cdot A^{-1} B^T X. \quad (3.1.4)$$

We can therefore choose

$$(\varphi_{\alpha, \beta_1}, \dots, \varphi_{\alpha, \beta_t}) = a \cdot A^{-1} B^T.$$

And one can easily observe that each  $\varphi_{\alpha, \beta_i} \in C_b^\infty(\mathbb{R}^N)$ , for  $i = 1, \dots, t$ .  $\square$

Throughout this chapter we assume that the UFG condition is satisfied. For  $\alpha \in \mathcal{A}$ , let us define the differential operator  $V_\alpha$  as follows:

$$V_\alpha = \text{Identity}, \quad \text{if } \alpha = \emptyset$$

and

$$V_\alpha = V_{\alpha_{i_1}} \cdots V_{\alpha_{i_k}}, \quad \text{if } \alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}).$$

We define the semi-norm  $\|\cdot\|_{V,n}$ ,  $n \geq 1$  on  $C_0^\infty(\mathbb{R}^N, \mathbb{R})$  by

$$\|f\|_{V,n} = \sum_{k=1}^n \sum_{\substack{\alpha_{i_1}, \dots, \alpha_{i_k} \\ \|\alpha_{i_1} * \dots * \alpha_{i_k}\| = n}} \|V_{[\alpha_{i_1}]} \cdots V_{[\alpha_{i_k}]} f\|_\infty.$$

We also define the semi-group of linear operators  $\{P_t\}_{t \in [0, \infty)}$  by

$$(P_t f)(x) = \mathbb{E}[f(X_t^x)], \quad t \in [0, \infty), \quad f \in C_b^\infty(\mathbb{R}^N).$$

**Definition 3.1.4.** Let  $m \geq 1$  be an integer. A family of random variables  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  is said to be  $m$ -moment similar if

$$Z_{(\alpha_0)} = 1, \\ \mathbb{E}[|Z_\alpha|^n] < \infty \quad \text{for any } \alpha \in \mathcal{A}_0 \quad \text{and } n \geq 1$$

and if

$$\mathbb{E}[Z_{\alpha_1} \cdots Z_{\alpha_k}] = \mathbb{E}[B^{\circ\alpha_1}(1) \cdots B^{\circ\alpha_k}(1)]$$

for all  $k = 1, \dots, m$  and  $\alpha_1, \dots, \alpha_k \in \mathcal{A}_0$  which satisfy  $\|\alpha_1\| + \dots + \|\alpha_k\| \leq m$  where  $B^{\circ\alpha_i}(1)$ ,  $i = 1, \dots, k$  are defined as in Definition 2.2.3.

We give some examples to illustrate the previous definition. These examples were introduced in Shunli (February 2003) and in Ninomiya (2003b).

**Example 3.1.5.** (1-dimensional 3-moment similar family )

Let  $\eta$  be a random variable defined by:

$$\mathbb{P}(\eta = 1) = \frac{1}{2}, \quad \mathbb{P}(\eta = -1) = \frac{1}{2}.$$

Then we can easily see that,  $\mathbb{E}[\eta^k] = 0 < \infty$  for all odd  $k$  and  $\mathbb{E}[\eta^k] = 1 < \infty$  for all  $k$  even. We define the family of random variables  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  as follows:

$$Z_{(\alpha_1)} = \eta, \quad Z_{(\alpha_0)} = 1, \quad Z_{(\alpha_1, \alpha_1)} = \frac{1}{2}\eta^2 \\ Z_\alpha = 0 \quad \text{for } \|\alpha\| \geq 3.$$

Then  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  is a 3-moment similar family of random variables.

**Example 3.1.6.** (1-dimensional 5-moment similar family 1)

Let  $d = 1$  and let  $\eta$  be a random variable such that

$$\mathbb{P}(\eta = 0) = \frac{1}{2}, \quad \mathbb{P}\left(\eta = \pm\sqrt{2 \pm \sqrt{2}}\right) = \frac{1}{8}.$$

By symmetry, one can easily show that

$$\mathbb{E}[\eta^k] = 0 < \infty$$

for all odd  $k$ .

$$\begin{aligned} \mathbb{E}[\eta^4] &= 0 * \frac{1}{2} + \frac{1}{8} \left( \left( \sqrt{2 + \sqrt{2}} \right)^4 + \left( -\sqrt{2 + \sqrt{2}} \right)^4 \right. \\ &\quad \left. + \left( \sqrt{2 - \sqrt{2}} \right)^4 + \left( -\sqrt{2 - \sqrt{2}} \right)^4 \right) \\ &= \frac{1}{8} \left( 2 * (2 + \sqrt{2})^2 + 2 * (2 - \sqrt{2})^2 \right) \\ &= \frac{1}{4} (6 + 4 * \sqrt{2} + 6 - 4 * \sqrt{2}) \\ &= 3. \end{aligned}$$

In the same way, we can show that  $\mathbb{E}[\eta^n] < \infty$  for all  $n$  even.

Now let us define the family of random variables  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  as follows:

$$\begin{aligned} Z_{(\alpha_1)} &= \eta, & Z_{(\alpha_0)} &= 1, & Z_{(\alpha_1, \alpha_1)} &= \frac{1}{2}\eta^2 \\ Z_{(\alpha_0, \alpha_1)} &= Z_{(\alpha_1, \alpha_0)} = \frac{1}{2}\eta, & Z_{(\alpha_1, \alpha_1, \alpha_1)} &= \frac{1}{6}\eta^3, & Z_{(\alpha_1, \alpha_1, \alpha_1, \alpha_1)} &= \frac{1}{8} \\ Z_{(\alpha_0, \alpha_1, \alpha_1)} &= Z_{(\alpha_1, \alpha_1, \alpha_0)} = \frac{1}{4}, & Z_{(\alpha_0, \alpha_0)} &= \frac{1}{2}, & & \\ & & Z_\alpha &= 0 & \text{in the other cases.} \end{aligned}$$

Then the family of random variables  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  is a 1-dimensional 5-moment similar family.

**Example 3.1.7.** (1-dimensional 5-moment similar family 2) Here again we consider  $d = 1$ , let  $\eta$  be a random variable verifying

$$\mathbb{P}(\eta = 0) = \frac{2}{3}, \quad \mathbb{P}(\eta = \pm\sqrt{3}) = \frac{1}{6}.$$

As in the previous example, we can easily verify that  $\mathbb{E}[\eta^k] < \infty$  for all  $k \geq 1$ . Let the family of random variables  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  be defined in the same way as in Example 3.1.6. Then  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  is also a 1-dimensional 5-moment similar family.

**Example 3.1.8.** (2-dimensional 5-moment similar family 1) Let  $d = 2$  and let  $\eta_1, \eta_2$  and  $\eta_{12}$  be independent random variables defined by

$$\mathbb{P}(\eta_{12} = \pm 1) = \frac{1}{2}, \quad \mathbb{P}(\eta_i = 0) = \frac{2}{3} \quad \text{and} \quad \mathbb{P}(\eta_i = \pm\sqrt{3}) = \frac{1}{6}.$$

We then define the family of random variables  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  as follows:

$$\begin{aligned} Z_{(\alpha_0)} &= 1, & Z_{(\alpha_1)} &= \eta_1, & Z_{(\alpha_2)} &= \eta_2, \\ Z_{(\alpha_1, \alpha_2)} &= \frac{1}{2}(\eta_1\eta_2 + \eta_{12}), & Z_{(\alpha_2, \alpha_1)} &= \frac{1}{2}(\eta_1\eta_2 - \eta_{12}), \\ Z_{(\alpha_i, \alpha_i)} &= \frac{1}{2}\eta_i^2, & Z_{(\alpha_i, \alpha_0)} &= Z_{(\alpha_0, \alpha_i)} = \frac{1}{2}\eta_i, & i \in \{1, 2\}, \\ Z_{(\alpha_i, \alpha_i, \alpha_i)} &= \frac{1}{6}\eta_i^3, & i \in \{1, 2\}, \\ Z_{(\alpha_1, \alpha_1, \alpha_2)} &= Z_{(\alpha_2, \alpha_1, \alpha_1)} = \frac{1}{4}\eta_2, \\ Z_{(\alpha_2, \alpha_2, \alpha_1)} &= Z_{(\alpha_1, \alpha_2, \alpha_2)} = \frac{1}{4}\eta_1, \\ Z_{(\alpha_0, \alpha_0)} &= \frac{1}{2}, \\ Z_{(\alpha_i, \alpha_i, \alpha_j, \alpha_j)} &= \frac{1}{8}, & Z_{(\alpha_0, \alpha_i, \alpha_i)} &= Z_{(\alpha_i, \alpha_i, \alpha_0)} = \frac{1}{4}, & i \in \{1, 2\}, \\ & & Z_\alpha &= 0 & \text{otherwise.} \end{aligned}$$

Then  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  is a 2-dimensional 5-moment similar family.

In a more general point of view the following is an example of 5-moment similar family for an arbitrary dimension  $d$ . This is taken from Kusuoka (2001):

**Example 3.1.9.** *let  $d \geq 1$  be a fixed dimension, let  $\eta_i, i = 1, \dots, d$  and  $\eta_{ij}, 1 \leq i < j \leq d$  be independent random variables defined by*

$$\mathbb{P}(\eta_{ij} = \pm 1) = \frac{1}{2}, \quad \mathbb{P}(\eta_i = 0) = \frac{1}{2} \quad \text{and} \quad \mathbb{P}\left(\eta_i = \pm \sqrt{2 \pm \sqrt{2}}\right) = \frac{1}{8}.$$

*As in Example 3.1.6, we can easily check that  $\mathbb{E}[\eta^k] < \infty$  for all  $k \geq 1$ . The family of random variables  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  in the following way:*

1. if  $\|\alpha\| = \|(i)\| = 1$ ,

$$Z_{(\alpha_i)} = \eta_i, \quad i = 1, \dots, d$$

2. if  $\|\alpha\| = 2$ ,

$$\begin{aligned} Z_{(\alpha_0)} &= 1, \\ Z_{(\alpha_i, \alpha_j)} &= \begin{cases} \frac{1}{2}(\eta_i \eta_j + \eta_{ij}), & 1 \leq i < j \leq d, \\ \frac{1}{2}(\eta_i \eta_j - \eta_{ij}), & 1 \leq j < i \leq d, \\ \frac{1}{2}\eta_i \eta_j, & 1 \leq i = j \leq d. \end{cases} \end{aligned}$$

3. if  $\|\alpha\| = 3$ ,

$$\begin{aligned} Z_{(\alpha_i, \alpha_0)} &= Z_{(\alpha_0, \alpha_i)} = \frac{1}{2}\eta_i, \quad Z_{(\alpha_i, \alpha_i, \alpha_i)} = \frac{1}{6}\eta_i^3, \quad 1 \leq i \leq d \\ Z_{(\alpha_i, \alpha_i, \alpha_j)} &= Z_{(\alpha_j, \alpha_i, \alpha_i)} = \frac{1}{4}\eta_i, \quad Z_{(\alpha_i, \alpha_j, \alpha_i)} = 0, \quad 1 \leq i \neq j \leq d, \end{aligned}$$

and  $Z_\alpha = 0$  otherwise.

4. if  $\|\alpha\| = 4$ ,

$$\begin{aligned} Z_{(\alpha_i, \alpha_i, \alpha_j, \alpha_j)} &= \frac{1}{8}, \quad 1 \leq i, j \leq d, \\ Z_{(\alpha_0, \alpha_i, \alpha_i)} &= Z_{(\alpha_i, \alpha_i, \alpha_0)} = \frac{1}{4}, \quad 1 \leq i \leq d, \\ Z_{(\alpha_0, \alpha_0)} &= \frac{1}{2}, \end{aligned}$$

and  $Z_\alpha = 0$  otherwise.

5. if  $\|\alpha\| \geq 5$ ,

$$Z_\alpha = 0$$

*Then the family of random variables  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  is a  $d$ -dimensional 5-moment similar family.*

Considering the semi-group  $\{P_t\}_{t \in [0, \infty)}$  of linear operators defined previously, let  $H : \mathbb{R}^N \Rightarrow \mathbb{R}^N$  be the identity map given by  $H(x) = (x_1, \dots, x_N)$ ,  $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ . We introduce the operator  $\mathcal{Q}_{(s)}$  defined as follows:

**Definition 3.1.10.** Let  $m \in \mathbb{N}$  and  $\{Z_\alpha, \alpha \in \mathcal{A}_0\}$  be a  $m$ -moment similar family. For  $f \in C_b^\infty(\mathbb{R}^N)$  and  $0 \leq s < 1$ , we define

$$\mathcal{Q}_{(s)}f(x) = \mathbb{E} \left[ f \left( \sum_{k=0}^m \frac{1}{k!} \sum_{\substack{\alpha_1, \dots, \alpha_k \in \mathcal{A}_0 \\ \|\alpha_1\| + \dots + \|\alpha_k\| \leq m}} s^{\frac{1}{2}(\|\alpha_1\| + \dots + \|\alpha_k\|)} (P_{\alpha_1}^0 \dots P_{\alpha_k}^0) (V_{[\alpha_1]} \dots V_{[\alpha_k]} H)(x) \right) \right]$$

where  $V_{[\alpha_i]}$ ,  $i = 1, \dots, k$  are defined as in Definition 2.1.3 and

$$P_\alpha^0 = \frac{1}{|\alpha|} \sum_{k=1}^{|\alpha|} \frac{(-1)^{k+1}}{k} \sum_{\substack{\beta_1, \dots, \beta_k \in \mathcal{A}_0 \\ \beta_1 * \dots * \beta_k = \alpha}} Z_{\beta_1} \dots Z_{\beta_k}.$$

The operator  $\mathcal{Q}_{(s)}$  verifies: For any  $f \in C_b^\infty(\mathbb{R}^N)$ ,

- $f(x) \geq 0 \Rightarrow \mathcal{Q}_{(s)}f(x) = \mathbb{E}[f(\cdot)] \geq 0$
- $\|\mathcal{Q}_{(s)}\| = \sup_{\|f\|=1} \|\mathcal{Q}_{(s)}f\| \leq 1$ .

This means that,  $\mathcal{Q}_{(s)}$  is a Markov operator as defined in Horowitz (1974). We have the following result,

**Theorem 3.1.11.** Kusuoka (2001) Let  $m$  be an integer and suppose that a family of random variables  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  is  $m$ -moment similar. Let  $\mathcal{Q}_{(s)}$  be the Markov operator as defined above. Then, for any  $n \geq 1$ , there is a constant  $C > 0$  such that

$$\|P_s f - \mathcal{Q}_{(s)}f\|_\infty \leq C \left( \sum_{k=m+1}^{n(m+1)} s^{\frac{k}{2}} \|f\|_{V,k} + s^{(m+1)/2} \|\nabla f\|_\infty \right)$$

$s \in (0, 1], f \in C_b^\infty(\mathbb{R}^N)$ .

Considering Definition 3.1.10, the announced operator is constructed as follows:

**Definition 3.1.12.** Let  $0 = t_0^{(n)} < t_1^{(n)} < \dots < t_n^{(n)} = T$  be a partition of the interval  $[0, T]$  defined by  $t_k^{(n)} = k^\gamma n^{-\gamma} T$  where  $n \in \mathbb{N}$ ,  $\gamma$  is a positive constant, and let  $s_k = t_k^{(n)} - t_{k-1}^{(n)}$ . The operator  $\mathcal{Q}_T$  which approximates  $P_T$  is defined by

$$\mathcal{Q}_T f = \mathcal{Q}_{(s_n)} \mathcal{Q}_{(s_{n-1})} \dots \mathcal{Q}_{(s_1)} f \quad (3.1.5)$$

for a given function  $f$ . Where  $\mathcal{Q}_{(s_i)}$  for  $i = 1, \dots, n$ , is as in Definition 3.1.10.

The main result of this chapter is the following result due to Kusuoka (2001).

**Theorem 3.1.13.** *For  $f \in C_b^\infty(\mathbb{R}^N)$ , we define*

$$\epsilon(f) = \|P_T f - \mathcal{Q}_T f\|_\infty.$$

*Then, we have the following statements:*

(i) *If  $0 < \gamma < m - 1$ , there exists a constant  $C > 0$  such that*

$$\epsilon(f) \leq C n^{-\gamma/2} \|\nabla(f)\|_\infty \quad \text{for all } f \in C_b^\infty(\mathbb{R}^N) \text{ and } n \geq 1.$$

(ii) *If  $\gamma = m - 1$ , there exists a constant  $C > 0$  such that*

$$\epsilon(f) \leq C n^{-(m-1)/2} \log(n+1) \|\nabla(f)\|_\infty \quad \text{for all } f \in C_b^\infty(\mathbb{R}^N) \text{ and } n \geq 1.$$

(iii) *If  $\gamma > m - 1$ , there is a constant  $C > 0$  such that*

$$\epsilon(f) \leq C n^{-(m-1)/2} \|\nabla f\|_\infty \quad \text{for all } f \in C_b^\infty(\mathbb{R}^N) \text{ and } n \geq 1.$$

We refer the interested reader to Kusuoka (2001) for the proofs of Theorems 3.1.13 and 3.1.11.

## 3.2 Application to an Option Pricing Problem

In this section, we apply the Kusuoka approximation scheme presented in the previous section (Definition 3.1.12) to the computation of  $\mathbb{E}[f(X_T^x)]$  where  $X_T^x$  is the solution at time  $T$  of a given stochastic differential equation. We first consider a very simple case (example taken from Shunli (February 2003)), in order to show explicitly how to implement the Kusuoka approximation scheme. Subsequently, we present an application to a more general problem taken from Ninomiya (2003a), namely the pricing of an Asian call option.

### 3.2.1 Application to a Simple Case

We consider the trivial SDE

$$dX(t, x) = dB(t)$$

where  $(B(t))_t$  is a standard one-dimensional Brownian motion.

We want to use the Kusuoka scheme to compute  $\mathbb{E}[f(X_T^x)] = \mathbb{E}[f(x + B_T)]$ .

Notice that

$$X(t, x) = x + \int_0^t V_0(X_s^x) ds + \int_0^t V_1(X_s^x) \circ dB(s),$$

with  $V_0 = 0$  and  $V_1 = 1$ . In this example, we take  $m = 3$  and we consider the family of 3-moment similar random variables  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  defined in Example 3.1.5. According to Definition 3.1.10, the approximation operator  $\mathcal{Q}_{(s)}$  is constructed as follows:

$$\mathcal{Q}_{(s)}f(x) = \mathbb{E} \left[ f \left( \sum_{k=0}^5 P^k \right) \right],$$

where

$$P^k = \frac{1}{k!} \sum_{\substack{\alpha_1, \dots, \alpha_k \in \mathcal{A}_0 \\ \|\alpha_1\| + \dots + \|\alpha_k\| \leq 5}} s^{\frac{1}{2}(\|\alpha_1\| + \dots + \|\alpha_k\|)} (P_{\alpha_1}^0 \dots P_{\alpha_k}^0) (V_{[\alpha_1]} \dots V_{[\alpha_k]} H)(x).$$

More explicitly, we have

- $k=0$ ;

$$P^0 = x.$$

- $k=1$ ;  $\alpha_1 \in \{(1), (0), (1, 1), (0, 1), (1, 0), (1, 1, 1)\}$

$$\begin{aligned} P^1 &= s^{1/2} P_{(1)}^0 V_1(x) + s \left( P_{(0)}^0 V_0(x) + P_{(1,1)}^0 V_{[(1,1)]}(x) \right) \\ &\quad + s^{3/2} \left( P_{(0,1)}^0 V_{[(0,1)]}(x) + P_{(1,0)}^0 V_{[(1,0)]}(x) + P_{(1,1,1)}^0 V_{[(1,1,1)]}(x) \right) \end{aligned}$$

After little algebra, we have

$$P^1 = s^{1/2} \eta.$$

- $k=2$ ;  $(\alpha_1, \alpha_2) \in \{((1), (1)), ((0), (1)), ((1), (0)), ((1), (1, 1)), ((1, 1), (1))\}$

$$\begin{aligned} P^2 &= \frac{1}{2} s P_{(1)}^0 P_{(1)}^0 V_1 V_1(x) + \frac{1}{2} s^{3/2} \left( P_{(0)}^0 P_{(1)}^0 V_0 V_1(x) + P_{(1)}^0 P_{(0)}^0 V_1 V_0(x) \right) \\ &\quad + \frac{1}{2} s^{3/2} \left( P_{(1)}^0 P_{(1,1)}^0 V_1 V_{[(1,1)]}(x) + P_{(1,1)}^0 P_{(1)}^0 V_{[(1,1)]} V_1(x) \right) \end{aligned}$$

After little algebra, we have

$$P^2 = \frac{1}{2} s \eta^2 + \frac{1}{2} s^{3/2} \eta = \frac{1}{2} s + \frac{1}{2} s^{3/2} \eta$$

- $k=3$ ;  $(\alpha_1, \alpha_2, \alpha_3) = ((1), (1), (1))$

$$\begin{aligned} P^3 &= \frac{1}{6} s^{3/2} P_{(1)}^0 P_{(1)}^0 P_{(1)}^0 V_1 V_1 V_1(x) \\ &= \frac{1}{6} s^{3/2} \eta^3 = \frac{1}{6} s^{3/2} \eta. \end{aligned}$$

We then obtain that

$$\mathcal{Q}_{(s)}f(x) = \mathbb{E} \left[ f \left( x + s^{1/2}\eta + \frac{1}{2}s + \frac{2}{3}s^{3/2}\eta \right) \right].$$

Furthermore, we have

$$\mathcal{Q}_{(s_2)}\mathcal{Q}_{(s_1)}f(x) = \mathbb{E} \left[ f \left( x + s_1^{1/2}\eta_1 + \frac{1}{2}s_1 + \frac{2}{3}s_1^{3/2}\eta_1 + s_2^{1/2}\eta_2 + \frac{1}{2}s_2 + \frac{2}{3}s_2^{3/2}\eta_2 \right) \right],$$

where  $\eta_1, \eta_2$  are two independent random variables defined as in example 3.1.5. More generally, we have

$$\begin{aligned} \mathcal{Q}_{(s_n)}\mathcal{Q}_{(s_{n-1})}\cdots\mathcal{Q}_{(s_1)}f(x) &= \mathbb{E} \left[ f \left( x + s_1^{1/2}\eta_1 + \frac{1}{2}s_1 + \frac{2}{3}s_1^{3/2}\eta_1 + s_2^{1/2}\eta_2 \right. \right. \\ &\quad \left. \left. + \frac{1}{2}s_2 + \frac{2}{3}s_2^{3/2}\eta_2 + \cdots + s_n^{1/2}\eta_n + \frac{1}{2}s_n \right. \right. \\ &\quad \left. \left. + \frac{2}{3}s_n^{3/2}\eta_n \right) \right] \end{aligned}$$

here,  $\eta_1, \eta_2, \dots, \eta_n$  are independent and  $s_k = t_k - t_{k-1} = k^\gamma n^{-\gamma}T - (k-1)^\gamma n^{-\gamma}T$ .

### 3.2.2 The Pricing of an Asian Call Option

In Section 3.2.1, we have considered the simplest case for the implementation of the Kusuoka scheme. We consider an Asian call option of European type (exercised only at maturity) in the Black-Scholes market. More precisely, under the risk neutral probability, we have  $dX_1(t, x_1) = X_1(t, x_1)(r'dt + \sigma dB(t))$  where the interest rate of the risk-free asset  $r'$  and the volatility  $\sigma$  are both considered to be constants and  $B(t)$  is a one-dimensional standard Brownian motion. We want to calculate the price of this option with maturity  $T$  and strike price  $K$  written on the stock  $X_1$ . The payoff at  $T$  of this option is

$$\max \left( 0, \frac{1}{T} \int_0^T X_1(t, x_1) dt - K \right).$$

Let  $X_2(t, x_2) = x_2 + \int_0^t X_1(s, x_1) ds$  and  $X_t^x = (X_1(t, x_1), X_2(t, x_2))$ . The situation can be summarised as follows:

$$X_t^x = x + \int_0^t V_0(X_s^x) ds + \int_0^t V_1(X_s^x) \circ dB(s),$$

where  $x = (x_1, 0)$ ,  $V_0(y_1, y_2) = (ry_1, y_1)$ ,  $r = r' - \sigma^2/2$  and  $V_1(y_1, y_2) = (\sigma y_1, 0)$ . Then, the price of this call option is  $D \cdot \mathbb{E}[f(X_T^x)]$  where  $D$  is the discount factor given in this case by  $D = e^{-rT}$  and  $f(y_1, y_2) = \max(0, \frac{y_2}{T} - K)$ .



### 3.2.2.1 Implementation of the Kusuoka Approximation

We now apply the Kusuoka approximation presented in Definition 3.1.10 to calculate the price of this option. In this case,  $d = 1$  and  $\mathcal{A}_0 = \cup_{k=1}^{\infty} \{0, 1\}^k$ . First of all, we have to construct an  $m$ -moment similar family of random variables. Here we consider  $m = 5$  and let  $\{Z_{\alpha}; \alpha \in \mathcal{A}_0\}$  be the 5-moment similar family defined in Example 3.1.6. Following the Definition 3.1.10, the approximation operator  $\mathcal{Q}_{(s)}$  is constructed as follows:

$$(\mathcal{Q}_{(s)}f)(x) = \mathbb{E}[f(G(s, \eta, x))]$$

where  $G$  is defined by

$$G(s, \eta, x) = \sum_{k=0}^5 \frac{1}{k!} \sum_{\substack{\alpha_1, \dots, \alpha_k \in \mathcal{A}_0 \\ \|\alpha_1\| + \dots + \|\alpha_k\| \leq 5}} s^{\frac{1}{2}(\|\alpha_1\| + \dots + \|\alpha_k\|)} (P_{\alpha_1}^0 \dots P_{\alpha_k}^0) (V_{[\alpha_1]} \dots V_{[\alpha_k]} H)(x) \quad (3.2.1)$$

with  $P_{\alpha}^0$  defined in Definition 3.1.10. Let us evaluate explicitly  $G(s, \eta, x)$ . We set

$$P^k = \sum_{\substack{\alpha_1, \dots, \alpha_k \in \mathcal{A}_0 \\ \|\alpha_1\| + \dots + \|\alpha_k\| \leq 5}} s^{\frac{1}{2}(\|\alpha_1\| + \dots + \|\alpha_k\|)} (P_{\alpha_1}^0 \dots P_{\alpha_k}^0) (V_{[\alpha_1]} \dots V_{[\alpha_k]} H)(x)$$

for  $k = 0, 1, \dots, 5$  and

$$P_{\alpha_1, \dots, \alpha_k}^k = s^{\frac{1}{2}(\|\alpha_1\| + \dots + \|\alpha_k\|)} (P_{\alpha_1}^0 \dots P_{\alpha_k}^0) (V_{[\alpha_1]} \dots V_{[\alpha_k]} H)(x),$$

where  $\alpha_1, \dots, \alpha_k \in \mathcal{A}_0$  and  $\|\alpha_1\| + \dots + \|\alpha_k\| \leq 5$ .

For  $k = 0$ , we have that  $P^0 = (x_1, x_2)$ .

For  $k = 1$ , we have that  $\alpha_1 \in \{(1); (0); (1, 1); (0, 1); (1, 0); (1, 1, 1); (0, 0); (0, 1, 1); (1, 0, 1); (1, 1, 0); (1, 1, 1, 1); (0, 0, 1); (1, 0, 0); (0, 1, 0); (0, 1, 1, 1); (1, 0, 1, 1); (1, 1, 0, 1); (1, 1, 1, 0); (1, 1, 1, 1, 1)\}$ . We obtain that

$$\begin{aligned} P_{(1)}^1 &= s^{1/2} Z_{(1)} (V_{[(1)]} H)(x) \\ &= s^{1/2} \eta V_1(x) \\ P_{(1)}^1 &= s^{1/2} \eta (\sigma x_1, 0). \end{aligned}$$

$$\begin{aligned} P_{(1,0,1,1)}^1 &= \frac{1}{4} s^{5/2} \left( -\frac{1}{2} Z_{(1)} Z_{(0,1,1)} - \frac{1}{2} Z_{(1,0)} Z_{(1,1)} - \frac{1}{2} Z_{(1,0,1)} Z_{(1)} \right. \\ &\quad \left. + \frac{1}{3} Z_{(1)} Z_{(0)} Z_{(1,1)} + \frac{1}{3} Z_{(1)} Z_{(0,1)} Z_{(1)} + \frac{1}{3} Z_{(1,0)} Z_{(1)} Z_{(1)} \right. \\ &\quad \left. - \frac{1}{4} Z_{(1)} Z_{(0)} Z_{(1)} Z_{(1)} \right) (V_{[(1,0,1,1)]} H)(x) \\ &= \frac{1}{4} s^{5/2} \left( -\frac{1}{8} \eta - \frac{1}{8} \eta^3 + \frac{1}{6} \eta^3 + \frac{1}{6} \eta^3 + \frac{1}{6} \eta^3 - \frac{1}{4} \eta^3 \right) (V_{[(1,0,1,1)]} H)(x) \\ &= \frac{1}{4} s^{5/2} \left( -\frac{1}{8} \eta + \frac{1}{8} \eta^3 \right) (V_{[(1,0,1,1)]} H)(x) \end{aligned}$$

$$\begin{aligned} (V_{[(1,0,1,1)]}H)(x) &= ([[[V_1, V_0], V_1], V_1]H)(x) \\ &= (0, \sigma^3 x_1) \end{aligned}$$

that is,

$$P_{(1,0,1,1)}^1 = \frac{1}{4}s^{5/2} \left( -\frac{1}{8}\eta + \frac{1}{8}\eta^3 \right) (0, \sigma^3 x_1).$$

Performing similar computations, we obtain

$$\begin{aligned} P_{(0)}^1 &= s(\mu x_1, x_1); \quad P_{(0,1,1)}^1 = s^2 \left( \frac{1}{12} - \frac{1}{18}\eta^2 \right) (0, \sigma^2 x_1) \\ P_{(0,1,0)}^1 &= \frac{1}{3}s^{5/2} \left( -\frac{1}{6}\eta \right) (0, \mu\sigma x_1); \quad P_{(1,0,0)}^1 = \frac{1}{3}s^{5/2} \left( -\frac{1}{6}\eta \right) (0, -\mu\sigma x_1) \\ P_{(0,1,1,1)}^1 &= \frac{1}{4}s^{5/2} \left( \frac{1}{24}\eta^3 - \frac{1}{8}\eta \right) (0, -\sigma^3 x_1); \quad \text{and} \quad P_{\alpha_k}^1 = 0 \quad \text{otherwise.} \end{aligned}$$

So we can conclude that

$$\begin{aligned} P^1 &= s^{1/2}\eta(\sigma x_1, 0) + \frac{1}{4}s^{5/2} \left( -\frac{1}{8}\eta + \frac{1}{8}\eta^3 \right) (0, -\sigma^3 x_1) + s(\mu x_1, x_1) \\ &\quad + s^2 \left( \frac{1}{12} - \frac{1}{18}\eta^2 \right) (0, \sigma^2 x_1) + \frac{1}{3}s^{5/2} \left( -\frac{1}{6}\eta \right) (0, \mu\sigma x_1) \\ &\quad + \frac{1}{3}s^{5/2} \left( -\frac{1}{6}\eta \right) (0, -\mu\sigma x_1) + \frac{1}{4}s^{5/2} \left( \frac{1}{24}\eta^3 - \frac{1}{8}\eta \right) (0, \sigma^3 x_1) \\ &\quad + \frac{1}{3}s^2 \left( -\frac{1}{6}\eta^2 \right) (0, -\sigma^2 x_1). \end{aligned}$$

For  $k = 2$ ,

$$\begin{aligned} P^2 &= s\eta^2(\sigma^2 x_1, 0) + s^{3/2}\eta(\mu\sigma x_1, \sigma x_1) + s^{3/2}(\mu^2 x_1, \mu x_1) \\ &\quad + s^{3/2}\eta(\mu\sigma x_1, 0) + s^{5/2} \left( -\frac{1}{18}\eta^3 \right) (0, -\sigma^3 x_1) \\ &\quad + s^{5/2} \left( \frac{1}{12}\eta - \frac{1}{18}\eta^3 \right) (0, \sigma^3 x_1). \end{aligned}$$

For  $k = 3$ ,

$$\begin{aligned} P^3 &= s^{3/2}\eta^3(\sigma^3 x_1, 0) + s^2\eta^2(\mu\sigma^2 x_1, \sigma^2 x_1) + s^2\eta^2(\mu\sigma^2 x_1, 0) \\ &\quad + s^2\eta^2(\mu\sigma^2 x_1, 0) + s^{5/2}\eta(\mu^2\sigma x_1, \sigma x_1) + s^{5/2}\eta(\mu^2\sigma x_1, \sigma x_1) \\ &\quad + s^{5/2}\eta(\mu^2\sigma x_1, 0). \end{aligned}$$

For  $k = 4$ ,

$$\begin{aligned} P^4 &= s^2\eta^4(\sigma^4 x_1, 0) + s^{5/2}\eta^3(\mu\sigma^3 x_1, \sigma^3 x_1) + s^{5/2}\eta^3(\mu\sigma^3 x_1, 0) \\ &\quad + s^{5/2}\eta^3(\mu\sigma^3 x_1, 0) + s^{5/2}\eta^3(\mu\sigma^3 x_1, 0). \end{aligned}$$

For  $k = 5$ ,

$$P^5 = s^{5/2} \eta^5 (\sigma^5 x_1, 0).$$

Finally, we obtain

$$\begin{aligned} G(s, \eta, x) = & (x_1, x_2) + s^{1/2} \eta (\sigma x_1, 0) + s^{5/2} \left( \frac{\eta}{32} - \frac{\eta^3}{32} \right) (0, \sigma^3 x_1) \\ & + s (\mu x_1, x_1) + s^2 \left( \frac{1}{12} - \frac{\eta^2}{18} \right) (0, \sigma^2 x_1) \\ & + s^{5/2} \left( \frac{\eta^3}{96} - \frac{\eta}{32} \right) (0, \sigma^3 x_1) + s^2 \left( \frac{\eta^2}{18} \right) (0, \sigma^2 x_1) \\ & + s \left( \frac{\eta^2}{2} \right) (\sigma^2 x_1, 0) + s^{3/2} \left( \frac{\eta}{2} \right) (\mu \sigma x_1, \sigma x_1) \\ & + \frac{1}{2} s^2 (\mu^2 x_1, \mu x_1) + s^{3/2} \left( \frac{\eta}{2} \right) (\mu \sigma x_1, 0) \\ & + s^{5/2} \left( \frac{\eta^3}{36} \right) (0, \sigma^3 x_1) + s^{5/2} \left( \frac{1}{24} \eta - \frac{1}{36} \eta^3 \right) (0, \sigma^3 x_1) \\ & + s^{3/2} \left( \frac{\eta^3}{6} \right) (\sigma^3 x_1, 0) + s^2 \left( \frac{\eta^2}{6} \right) (\mu \sigma^2 x_1, \sigma^2 x_1) \\ & + s^2 \left( \frac{\eta^2}{3} \right) (\mu \sigma^2 x_1, 0) + s^{5/2} \left( \frac{\eta}{3} \right) (\mu^2 \sigma x_1, \mu * \sigma x_1) \\ & + s^{5/2} \left( \frac{\eta}{6} \right) (\mu^2 \sigma x_1, 0) + s^2 \left( \frac{\eta^4}{24} \right) (\sigma^4 x_1, 0) \\ & + s^{5/2} \left( \frac{\eta^3}{24} \right) (\mu \sigma^3 x_1, \sigma^3 x_1) + s^{5/2} \left( \frac{\eta^3}{8} \right) (\mu \sigma^3 x_1, 0) \\ & + s^{5/2} \left( \frac{\eta^5}{120} \right) (\sigma^5 x_1, 0). \end{aligned}$$

We now implement the calculation of  $\mathbb{E}[f(X_t^x)]$  by computing

$\mathcal{Q}_{(s_n)} \mathcal{Q}_{(s_{n-1})} \cdots \mathcal{Q}_{(s_1)} f$ , where

$$(\mathcal{Q}_{(s_i)} f)(x) = \mathbb{E}[f(G(s_i, \eta, x))], \quad (3.2.2)$$

for  $i = 1, \dots, n$ .

Let  $(w_0, w_1, w_2, w_3, w_4) = (0, \sqrt{2 + \sqrt{2}}, -\sqrt{2 + \sqrt{2}}, \sqrt{2 - \sqrt{2}}, -\sqrt{2 - \sqrt{2}})$  and  $p_i = P(\eta = w_i)$ . We then compute  $(\mathcal{Q}_{(s_i)} f)(x) = \mathbb{E}[f(G(s_i, \eta, x))]$  as follows:

$$\mathbb{E}[f(G(s_i, \eta, x))] = \sum_{k(i)=0}^4 p_{k(i)} f(G(s_i, w_{k(i)}, x)) \quad (3.2.3)$$

that is, considering equation (3.2.2) and equation (3.2.3),

$$\begin{aligned}
 (\mathcal{Q}_{(s_2)} \mathcal{Q}_{(s_1)} f)(x) &= \sum_{k(2)=0}^4 p_{k(2)} (\mathcal{Q}_{(s_1)} f)(G(s_2, w_{k(2)}, x)) \\
 &= \sum_{k(2)=0}^4 p_{k(2)} \sum_{k(1)=0}^4 p_{k(1)} f(G(s_1, w_{k(1)}, G(s_2, w_{k(2)}, x))) \\
 &= \sum_{k(2)=0}^4 \sum_{k(1)=0}^4 p_{k(2)} \cdot p_{k(1)} f(G(s_1, w_{k(1)}, G(s_2, w_{k(2)}, x))).
 \end{aligned}$$

By recurrence, we obtain

$$\begin{aligned}
 (\mathcal{Q}_{(s_n)} \mathcal{Q}_{(s_{n-1})} \cdots \mathcal{Q}_{(s_1)} f)(x) &= \sum_{k(n)=0}^4 \sum_{k(n-1)=0}^4 \cdots \sum_{k(1)=0}^4 \left( \prod_{j=1}^n p_{k(j)} \right) f(G(s_1, w_{k(1)}, \\
 &\quad G(s_2, w_{k(2)}, G(\cdots, G(s_n, w_{k(n)}, x) \cdots))).
 \end{aligned}$$

### 3.2.2.2 Numerical Results

We numerically compare the Kusuoka scheme with the traditional Euler-Maruyama scheme.

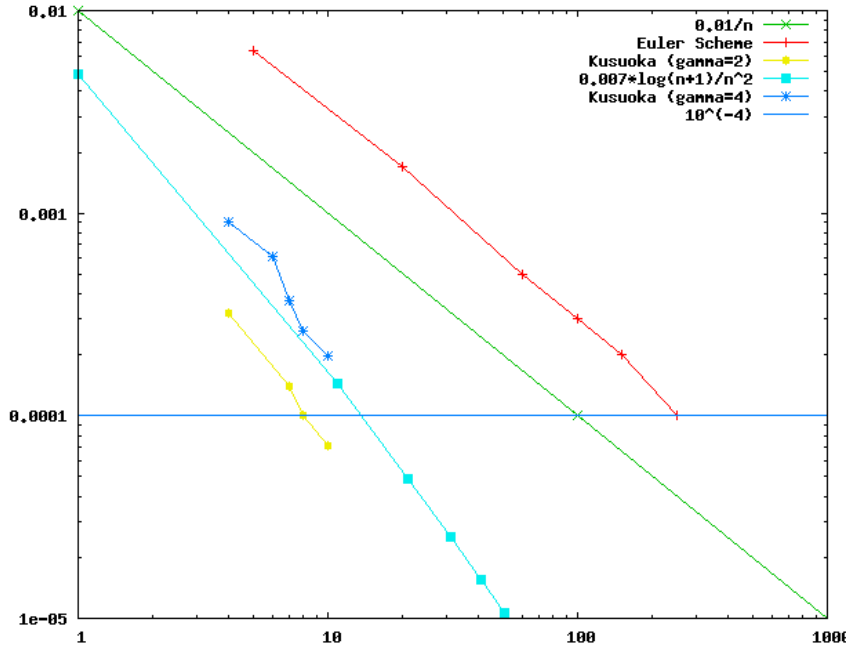
In this experiment, we consider

$$\mathbb{E}[\max(X_2(T, x)/T - K, 0)] = 1.7780997 \times 10^{-2}$$

which is taken from Ninomiya (2003a) as our benchmark (obtained with the Monte Carlo method with more than  $10^8$  sample point).

Figure 3.1 shows the relation between  $n$  the number of partitions and the approximation error that comes from the discretization. The plots are done on a log-log scale which enable us to obtain linear plots and to easily interpret the results. We set, respectively,  $\gamma = 2$  and  $\gamma = 4$  and the numbers of sample points for the Monte Carlo calculation is  $M = 10^4$ . The results are the following:

- (1) In the case of the Kusuoka approximation with  $\gamma = 2$ ,  $n = 8$  is enough to achieve  $10^{-4}$  accuracy and for the Euler-Maruyama approximation with  $10^4$  sample size Monte-Carlo,  $n$  must be greater than 2500.
- (2) The approximation error of the Kusuoka approximation is almost consistent with Theorem 3.1.13 and that of the Euler approximation is proportional to  $n^{-1}$ , also consistent with the theoretical result.
- (3) It takes  $2.462 \times 10^4$  seconds to compute the result using the Euler scheme and only 19.73 seconds when using the Kusuoka approximation. Therefore, we can affirm that in this experiment the Kusuoka scheme has achieved 1245 times faster calculation than the traditional Euler approximation.



**Figure 3.1:** Comparison of errors of the Kusuoka and Euler schemes.

The approximation considered in our example is realized by a relatively small number of points, so we were able to proceed with the calculation by hand. That is to say, we do not need any partial sampling scheme such as Monte Carlo or Quasi-Monte Carlo methods for our new approximation scheme.

In general, the number of points used in the approximation is very large especially when the dimension of the process  $X_t^x$  under consideration is greater than one or the length of the period to be simulated ( $T$ ) becomes longer and in such cases, we need a partial sampling from a finite set of points. Ninomiya (2003b) reported that by using the tree-based branching algorithm (TBBA) in the Kusuoka approximation, we can achieve a several hundred times faster calculation. However, with the Kusuoka scheme, there are still some problems such as:

- it is not easy to construct moment similar families of random variables.
- the construction of the approximation operator  $\mathcal{Q}_{(s)}$  is very difficult, especially when the dimension of the Brownian motion under consideration is high.

In the next chapters, we present other approximation schemes which can be viewed as a generalisation of the Kusuoka scheme.

## Chapter 4

# Cubature Formula on Wiener Space

It is well known that in finite dimension, cubature formulas provide good approximative values for integrals with respect to a given measure. The theory of cubature formulas in a finite dimension was recently extended by Lyons and Victoir, 2004 to provide approximative values for integrals defined on Wiener space. This provides another weak approximation scheme to stochastic differential equations which is similar to the Kusuoka approximation presented in Chapter 3. Cubature formulas on Wiener space are based on the basic observation that any diffusion can be constructed up to given high-order asymptotic, by a superposition of iterated Stratonovich integrals in well-specified directions. Hence it is sufficient to obtain Cubature results for those iterated Stratonovich integrals in order to obtain the results for any diffusion. The cubature method is very attractive according to the mathematical point of view because it combines results from different areas such as Numerical Analysis (classical cubature formula), Algebra and Differential Geometry, together with Stochastic Analysis.

In this chapter we outline the main results of this new approximation scheme. For more detail on the topic we refer the interested reader to Lyons and Victoir, 2004; Bayer and Teichmann, 2008 and references therein. We start the presentation of the method with a look at the classical cubature formulas in Section 4.1. In Section 4.2 we present the extension of the classical cubature formulas to Wiener space by Lyons and Victoir. The algorithm of this new approximation scheme is presented in Section 4.3. We distinguish between the case where the function  $f$  in consideration is smooth and when it is only Lipschitz continuous. We close this chapter by looking at some examples of the construction of cubature formula on Wiener space.

## 4.1 Classical Cubature Formula

A cubature formula is a classical technique of numerical integration. It provides approximate values for integrals over finite dimensional spaces with respect to a given measure. Let us recall that the support of a positive measure  $\mu$  on  $\mathbb{R}^d$ , denoted by  $\text{supp } \mu$ , is the complement of the biggest open set  $O \in \mathbb{R}^d$  with  $\mu(O) = 0$ . We denote by  $\mathbb{R}_m[X_1, \dots, X_d]$  the space of polynomials  $p$  in  $d$  variables and of degree less than or equal to  $m$ .

**Definition 4.1.1.** *Given a positive measure  $\mu$  on  $\mathbb{R}^d$ , and a natural number  $m \in \mathbb{N}$  such that all moments of  $\mu$  up to order  $m$  exist, i.e.*

*$\int_{\mathbb{R}^d} \|x\|^k \mu(dx) < \infty$ ,  $k = 0, \dots, m$ . A finite sequence of points  $x_1, \dots, x_n$  in the support of  $\mu$ , and positive weights  $\lambda_1, \dots, \lambda_n$  is said to define a cubature formula of degree  $m$  with respect to the measure  $\mu$  if, and only if,*

$$\int_{\mathbb{R}^d} p(x) \mu(dx) = \sum_{i=1}^n \lambda_i p(x_i) \quad (4.1.1)$$

*for all polynomials  $p \in \mathbb{R}_m[X_1, \dots, X_d]$ .*

**Remark 4.1.2.** *When  $d = 1$ , one talks about a quadrature formula rather than a cubature formula.*

Even though construction of cubature formulas is a non trivial task, especially in higher dimensions, there is a very general existence result which is given by

**Theorem 4.1.3.** *Let  $m$  be a positive integer and  $\mu$  a positive measure on  $\mathbb{R}^d$  with the property that all moments up to order  $m$  exist. Then there is an integer  $n$  with  $1 < n \leq \dim \mathbb{R}_m[X_1, \dots, X_d]$ , a sequence of points  $x_1, \dots, x_n \in \mathbb{R}^d$  and positive weights  $\lambda_1, \dots, \lambda_n$  such that for all  $p \in \mathbb{R}_m[X_1, \dots, X_d]$ , the cubature relation (4.1.1) holds.*

Theorem 4.1.3, also known as the Tchakaloff Theorem, was first proved by Tchakaloff in 1957 where he considered compactly supported Borel measures and later on, the proof was extended to more general cases (non-compactly supported measures) by many other authors among whom was Putinar (1997). It states that the cubature formula exists and moreover, that the formula can be constructed with the number of points ( $n$  in our case) being less than, or equal to  $\dim \mathbb{R}_m[x_1, \dots, x_d]$ .

Given a cubature formula as in Definition 4.1.1 and a function  $f$ , one would then use expression  $\sum_{i=1}^n \lambda_i f(x_i)$  to approximate  $\int_{\mathbb{R}^d} f(x) \mu(dx)$ . The accuracy of such approximation depends on whether or not  $f$  can be well approximated by polynomials. This shows, considering Taylor's formula, that, for smooth functions, a cubature formula is an efficient method for numerical integration. (Note that  $f$  only needs to be defined on  $\text{supp } \mu$  for the integral

$\int f d\mu$  to make sense). Therefore, we require that all points of a cubature formula lie in  $\text{supp } \mu$ , (see Bayer and Teichmann, 2008 and references therein for more information on these points). Unfortunately, Theorem 4.1.3 is only an existence theorem, it does not provide any approach to construct such a cubature formula. Indeed, the construction of cubature formulas remains a highly non-trivial task, especially if the size should be close to the optimal size for the particular measure. We refer to Victoir (2004) for ideas about construction of cubature formulas in any given dimension.

In order to present the cubature on Wiener space, let us make the probabilistic representation of parabolic PDEs precise. Consider the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega \equiv C_0^0([0, T], \mathbb{R}^{d+1})$  is the Wiener space, that is the space of  $\mathbb{R}^d$ -valued continuous functions defined on  $[0, T]$  and starting at zero.  $\mathcal{F}$  is its Borel  $\sigma$ -algebra, and  $\mathbb{P}$  is the Wiener measure. For  $\omega \in \Omega$ , we set  $\omega^0(t) = t$ . We define the coordinate mapping process  $B_t^i(\omega) = \omega^i(t)$  for  $t \in [0, T]$ ,  $\omega \in \Omega$ . Notice that under the Wiener measure,  $B = (B_t^1, \dots, B_t^d)_{t \in [0, T]}$  is a  $d$ -dimensional standard Brownian motion. We fix  $B^0(t) = t$ . We want to compute

$$\mathbb{E}(f(X_T^x)) = \int_{\Omega} f(X_T^x(\omega)) \mathbb{P}(d\omega) \quad (4.1.2)$$

where  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is a given function and  $X_t^x$  is a stochastic process verifying

$$X_t^x = x + \sum_{j=0}^d \int_0^t V_j(X_s^x) \circ dB_s^j \quad (4.1.3)$$

$$V_j \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N). \quad (4.1.4)$$

Let us define the map,

$$\Phi_{t,x} : \Omega \rightarrow \mathbb{R}^N \quad (4.1.5)$$

$$\omega \mapsto X_t^x(\omega) \quad (4.1.6)$$

and consider  $u(t, x)$  to be the solution at time  $t$  of

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) &= Lu(t, x) \\ u(0, x) &= f(x) \end{cases} \quad (4.1.7)$$

where the operator  $L$  is defined as in Equation (3.1.10). Then,

$$u(T, x) = \int_{\Omega} f(\Phi_{T,x}(\omega)) \mathbb{P}(d\omega).$$

Hence, to approximate the solution of a parabolic partial differential equation, we have to approximate an integral over the Wiener space (an infinite dimensional space). Therefore, the question is whether the classical cubature



formula can be extended in order to help integrating (4.1.2). We see from the stochastic Taylor's formula, presented in Proposition 2.2.9, that  $f(X_t^x)$  is approximated by a sum of Stratonovich iterated integrals,

$$\sum_{\alpha \in \mathcal{A}(m)} V_{\alpha_1} \cdots V_{\alpha_k} f(x) \int_{0 < t_1 < \cdots < t_k < t} \circ dB_{t_1}^{\alpha_1} \circ \cdots \circ dB_{t_k}^{\alpha_k}.$$

In the same way as in the deterministic case, a smooth function  $g$  is approximated at a point  $x$  by a sum of polynomials,

$$g(x) \simeq \sum_{k \leq m} g^{(k)}(x_0) \frac{(x - x_0)^k}{k!}.$$

It now appears natural to give the definition of a cubature formula on Wiener space, this by replacing the polynomials we have in classical cubature formula by the Stratonovich iterated integrals and by fixing a positive measure on Wiener space.

## 4.2 Cubature Formula on Wiener Space

We denote by  $C_{0,bv}^0([0, T], \mathbb{R}^{d+1})$  the subset of  $C_0^0([0, T], \mathbb{R}^{d+1})$  consisting of bounded variation paths.

**Definition 4.2.1.** *Let  $m, n \in \mathbb{N}$ . A finite sequence of paths  $\omega_1, \dots, \omega_n \in C_{0,bv}^0([0, T], \mathbb{R}^{d+1})$  and positive weights  $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$  is said to define a cubature formula on Wiener space of degree  $m$  at a fixed time  $T$  if,*

$$\mathbb{E}(B^{\circ\alpha}(T)) = \sum_{i=1}^n \lambda_i \int_{0 < t_1 < \cdots < t_k < T} d\omega_i^{\alpha_{i_1}}(t_1) \cdots d\omega_i^{\alpha_{i_k}}(t_k) \quad (4.2.1)$$

for all  $\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m)$ .

Here,  $B^{\circ\alpha}(T)$  is given by Definition 2.2.3, as

$$B^{\circ\alpha}(T) = \int_{0 < t_1 < \cdots < t_k < T} \circ dB_{t_1}^{\alpha_{i_1}} \circ \cdots \circ dB_{t_k}^{\alpha_{i_k}} \quad (4.2.2)$$

Through all this chapter we will denote  $B^{\circ\alpha}(T)$  by  $B_T^{\circ\alpha}$ .

**Remark 4.2.2.** *Definition 4.2.1 also means that the expectation of the Stratonovich iterated integrals of degree less than or equal to  $m$  under the Wiener measure is the same as under the probability measure*

$$\mathbb{Q} = \sum_{j=1}^n \lambda_j \delta_{\omega_j}.$$

**Remark 4.2.3.** *The integrals on the right hand side of Equation (4.2.1) are well-defined because the cubature paths considered are required to be of bounded variation on  $[0, T]$ . Considering the definition of  $\omega^i(t)$ , given by  $B_t^i(\omega) = \omega^i(t)$ , we define  $\omega^\alpha(t)$  as:*

$$\omega^\alpha(t) := \int_{0 < t_1 < \dots < t_k < T} d\omega_i^{\alpha_{i_1}}(t_1) \cdots d\omega_i^{\alpha_{i_k}}(t_k).$$

*This may be view as the evaluation of the random variable  $B_T^\alpha(\omega)$  for a path  $\omega \in C_{0,bv}^0([0, T], \mathbb{R}^{d+1})$ . Then Equation (4.2.1) can be written as*

$$\mathbb{E}(B_T^\alpha) = \sum_{i=1}^n \lambda_i B_T^\alpha(\omega_i); \quad (4.2.3)$$

*which looks more similar to equation 4.1.1 in Definition 4.1.1.*

By the scaling property of the Brownian motion, it is enough to construct a cubature formula on Wiener space at time 1, as detailed in the following:

**Proposition 4.2.4.** *(Lyons and Victoir, 2004). Assume the paths  $\omega_1, \dots, \omega_n \in C_{0,bv}^0([0, 1], \mathbb{R}^{d+1})$  and positive weights  $\lambda_1, \dots, \lambda_n$  define a cubature formula on Wiener space of degree  $m$  at time 1. For  $i = 1, \dots, n$ , we define the paths  $\omega_{T,i}$  by  $\omega_{T,i}^j(t) = \sqrt{T}\omega_i^j(t/T)$ , for  $j = 1, \dots, d$ . Then the paths  $\omega_{T,i} \in C_{0,bv}^0([0, T], \mathbb{R}^{d+1})$  and the same weights  $\lambda_1, \dots, \lambda_n$  define a cubature formula on Wiener space of degree  $m$  at time  $T$ .*

The existence of a cubature formula on Wiener space was obtained by Lyons and Victoir, 2004 and it is stated in the following theorem:

**Theorem 4.2.5.** *(Lyons and Victoir, 2004). Let  $m$  be a natural number. Then one can find  $n$  paths of bounded variation  $\omega_1, \dots, \omega_n \in C_{0,bv}^0([0, T], \mathbb{R}^{d+1})$  and  $n$  positive weights  $\lambda_1, \dots, \lambda_n$ , with  $n \leq \text{card}\mathcal{A}(m)$  such that  $\omega_i$  and  $\lambda_i, i = 1, \dots, n$  define a cubature formula on Wiener space of degree  $m$  at time  $T$ .*

This result is an extension of Tchakaloff's theorem and we refer to Bayer (2004) for a proof.

### 4.3 Algorithm for Cubature Formulas on Wiener Space

Consider the paths  $\omega_1, \dots, \omega_n \in C_{0,bv}^0([0, 1], \mathbb{R}^{d+1})$  and positive weights  $\lambda_1, \dots, \lambda_n$ . Assume that  $\omega_i$  and  $\lambda_i, i = 1, \dots, n$  define a cubature formula on Wiener space of degree  $m$  at time 1 and define, for  $i = 1, \dots, n$ , the paths  $\omega_{T,i}$  in  $C_{0,bv}^0([0, 1], \mathbb{R}^{d+1})$  by  $\omega_{T,i}^0(t) = t$  and  $\omega_{T,i}^j(t) = \sqrt{T}\omega_i^j(t/T)$  for  $j = 1, \dots, d$ .

**Lemma 4.3.1.** *Considering the construction of the paths  $\omega_{T,i}$  and by the scaling property, we have that for all  $\alpha = (\alpha_{i_1}, \dots, \alpha_{i_n}) \in \mathcal{A}(m)$ ,*

$$\mathbb{E} \left( \int_{0 < t_1 < \dots < t_k < T} \circ dB_{t_1}^{\alpha_{i_1}} \circ \dots \circ dB_{t_k}^{\alpha_{i_k}} \right) = \mathbb{E}_{\mathbb{Q}_T} \left( \int_{0 < t_1 < \dots < t_k < T} \circ dB_{t_1}^{\alpha_{i_1}} \circ \dots \circ dB_{t_k}^{\alpha_{i_k}} \right) \quad (4.3.1)$$

where  $\mathbb{Q}_T$  is the probability measure defined by

$$\mathbb{Q}_T = \sum_{i=1}^n \lambda_i \delta_{\omega_{T,i}}.$$

*Proof.* By Proposition 4.2.4, the paths  $\omega_{T,i}$  and the positive weights  $\lambda_i$ ,  $i = 1, \dots, n$  define a cubature formula on Wiener space of degree  $m$  at time  $T$ , so

$$\begin{aligned} \mathbb{E} \left( \int_{0 < t_1 < \dots < t_k < T} \circ dB_{t_1}^{\alpha_{i_1}} \circ \dots \circ dB_{t_k}^{\alpha_{i_k}} \right) &= \sum_{i=1}^n \lambda_i \int_{0 < t_1 < \dots < t_k < T} d\omega_{T,i}^{\alpha_{i_1}}(t_1) \cdots d\omega_{T,i}^{\alpha_{i_k}}(t_k) \\ &= \mathbb{E}_{\mathbb{Q}_T} \left( \int_{0 < t_1 < \dots < t_k < T} \circ dB_{t_1}^{\alpha_{i_1}} \circ \dots \circ dB_{t_k}^{\alpha_{i_k}} \right). \end{aligned}$$

□

**Lemma 4.3.2.** *We adapt the stochastic Taylor's formula by considering the new probability measure  $\mathbb{Q}_T$  and we derive an upper bound for the remainder process  $R_m(T, x, f)$  as:*

$$\sup_x \mathbb{E}_{\mathbb{Q}_T} (|R_m(T, x, f)|) \leq C_{d,m,\mathbb{Q}_T} T^{(m+1)/2} \sup_{\alpha \in \mathcal{A}(m+2) \setminus \mathcal{A}(m)} \|V_{\alpha_{i_1}} \cdots V_{\alpha_{i_k}} f\|_{\infty} \quad (4.3.2)$$

where the constant  $C$  depends only on  $d$ ,  $m$  and  $\mathbb{Q}_T$ .

Since each path  $\omega_j \in C_{0,bv}^0([0, 1], \mathbb{R}^{d+1})$ , the following remark on Riemann-Stieltjes integrals will be useful in the proof of the Lemma 4.3.2.

**Remark 4.3.3.** *Let  $f$  be a smooth function and  $g$  be a function of bounded variation. Then*

$$\left| \int_0^t f(s) dg(s) \right| \leq \sup_{s \in [0,t]} |f(s)| |g|_{TV;[0,t]} \quad (4.3.3)$$

where  $|g|_{TV;[0,t]}$  denotes the total variation of  $g$  on the interval  $[0, t]$ , i.e.

$$|g|_{TV;[0,t]} = \sup_p \left\{ \sum_{i=1}^n |g(t_i) - g(t_{i-1})| \right\},$$

where  $p$  is the set of all partitions  $\{t_i\}_{i=0,\dots,n}$  of the interval  $[0, t]$ .

*Proof.* In the proof of Proposition 2.2.9, we showed that

$$R_m(T, x, f) = \sum_{\substack{\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m) \\ (\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_k}) \notin \mathcal{A}(m)}} \int_{0 < t_0 < t_1 < \dots < t_k < T} V_{\alpha_{i_0}} \cdots V_{\alpha_{i_k}} f(X_{t_0}^x) \\ \circ dB_{t_0}^{\alpha_{i_0}} \circ \dots \circ dB_{t_k}^{\alpha_{i_k}}.$$

Hence, by Lemma 4.3.1,  $\mathbb{E}_{\mathbb{Q}_T}(|R_m(T, x, f)|)$  is bounded by

$$\sum_{j=1}^n \lambda_j \sum_{\substack{\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m) \\ (\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_k}) \notin \mathcal{A}(m)}} \left| \int_{0 < t_0 < t_1 < \dots < t_k < T} V_{\alpha_{i_0}} \cdots V_{\alpha_{i_k}} f\left(X_{t_0}^x\left(\omega_{T,j}^{\alpha_{i_0}}\right)\right) \right. \\ \left. d\omega_{T,j}^{\alpha_{i_0}}(t_0) \cdots d\omega_{T,j}^{\alpha_{i_k}}(t_k) \right|.$$

For  $t, t_\ell \in [0, T]$ , we define

$$y = \frac{t}{T} \quad \text{and} \quad y_\ell = \frac{t_\ell}{T} \quad \text{for} \quad \ell = 0, \dots, k.$$

This implies,

$$d\omega_{T,j}^{\alpha_{i_\ell}}(t_\ell) = d\omega_{T,j}^{\alpha_{i_\ell}}(y_\ell T) = \sqrt{T} d\omega_j^{\alpha_{i_\ell}}(y_\ell) \quad (4.3.4)$$

for  $i_\ell = 1, \dots, d$  and

$$d\omega_{T,j}^{\alpha_0}(t_i) = dt_i = d(y_i T) = T dy_i.$$

By setting

$$h(t_0) = V_{\alpha_{i_0}} \cdots V_{\alpha_{i_k}} f\left(X_{t_0}^x\left(\omega_{T,j}^{\alpha_{i_0}}\right)\right),$$

we obtain,

$$\begin{aligned} & \int_{0 < t_0 < \dots < t_k < T} h(t_0) d\omega_{T,j}^{\alpha_{i_0}}(t_0) \cdots d\omega_{T,j}^{\alpha_{i_k}}(t_k) \\ &= \int_0^T \int_0^{t_k} \cdots \int_0^{t_1} h(t_0) d\omega_{T,j}^{\alpha_{i_0}}(t_0) \cdots d\omega_{T,j}^{\alpha_{i_k}}(t_k) \\ &= \int_0^1 \int_0^{t_k/T} \cdots \int_0^{t_1/T} h(y_0) \sqrt{T}^{\|\beta\|} d\omega_j^{\alpha_{i_0}}(y_0) \cdots d\omega_j^{\alpha_{i_k}}(y_k) \end{aligned}$$

where  $\beta = \alpha_{i_0} * \alpha = (\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_k})$ .

Let  $H = \left| \int_{0 < t_0 < \dots < t_k < T} h(t_0) d\omega_{T,j}^{\alpha_{i_0}}(t_0) \cdots d\omega_{T,j}^{\alpha_{i_k}}(t_k) \right|$ , considering the change of variable (4.3.4), we have

$$H = \sqrt{T}^{\|\beta\|} \left| \int_{0 < y_0 < \dots < y_k < 1} h(t_0) d\omega_j^{\alpha_{i_0}}(y_0) \cdots d\omega_j^{\alpha_{i_k}}(y_k) \right|.$$

Applying (4.3.3) successively to the integrals we get

$$\begin{aligned}
H &\leq \sqrt{T}^{\|\beta\|} \left| \omega_j^{\alpha_{i_k}} \right|_{TV;[0,1]} \sup_{y_k \in [0,1]} \left| \int_0^{y_k} \cdots \int_0^{y_1} h(y_0) d\omega_j^{\alpha_{i_0}}(y_0) \cdots d\omega_j^{\alpha_{i_{k-1}}}(y_{k-1}) \right| \\
&\leq \sqrt{T}^{\|\beta\|} \left| \omega_j^{\alpha_{i_k}} \right|_{TV;[0,1]} \sup_{y_k \in [0,1]} \left| \omega_j^{\alpha_{i_{k-1}}} \right|_{TV;[0,y_k]} \sup_{y_{k-1} \in [0,y_k]} \left| \int_0^{y_{k-1}} \cdots \int_0^{y_1} h(y_0) \right. \\
&\quad \left. d\omega_j^{\alpha_{i_0}}(y_0) \cdots d\omega_j^{\alpha_{i_{k-2}}}(y_{k-2}) \right| \\
&\leq \sqrt{T}^{\|\beta\|} \left| \omega_j^{\alpha_{i_k}} \right|_{TV;[0,1]} \left| \omega_j^{\alpha_{i_{k-1}}} \right|_{TV;[0,1]} \sup_{y_{k-1} \in [0,1]} \left| \int_0^{y_{k-1}} \cdots \int_0^{y_1} h(y_0) d\omega_j^{\alpha_{i_0}}(y_0) \right. \\
&\quad \left. \cdots d\omega_j^{\alpha_{i_{k-2}}}(y_{k-2}) \right|.
\end{aligned}$$

By iterating this procedure further, we obtain

$$H \leq \sqrt{T}^{\|\beta\|} \left| \omega_j^{\alpha_{i_k}} \right|_{TV;[0,1]} \left| \omega_j^{\alpha_{i_{k-1}}} \right|_{TV;[0,1]} \cdots \left| \omega_j^{\alpha_{i_0}} \right|_{TV;[0,1]} \sup_{y_0 \in [0,1]} |h(y_0)|$$

or

$$h(y_0) = V_{\alpha_{i_0}} \cdots V_{\alpha_{i_k}} f \left( X_{y_0}^x \left( \omega_{T,j}^{\alpha_{i_0}} \right) \right) \quad \text{and} \quad X_{y_0}^x \left( \omega_{T,j}^{\alpha_{i_0}} \right) \in \mathbb{R}^N$$

so,

$$\begin{aligned}
\sup_{y_0 \in [0,1]} |h(y_0)| &= \sup_{y_0 \in [0,1]} \left| V_{\alpha_{i_0}} \cdots V_{\alpha_{i_k}} f \left( X_{y_0}^x \left( \omega_{T,j}^{\alpha_{i_0}} \right) \right) \right| \\
&\leq \sup_{x \in \mathbb{R}^N} \left| V_{\alpha_{i_0}} \cdots V_{\alpha_{i_k}} f(x) \right|
\end{aligned}$$

which implies that,

$$\begin{aligned}
H &\leq \sqrt{T}^{\|\beta\|} \left| \omega_j^{\alpha_{i_k}} \right|_{TV;[0,1]} \left| \omega_j^{\alpha_{i_{k-1}}} \right|_{TV;[0,1]} \cdots \left| \omega_j^{\alpha_{i_0}} \right|_{TV;[0,1]} \sup_{x \in \mathbb{R}^N} \left| V_{\alpha_{i_0}} \cdots V_{\alpha_{i_k}} f(x) \right| \\
&= \sqrt{T}^{\|\beta\|} \left| \omega_j^{\alpha_{i_k}} \right|_{TV;[0,1]} \left| \omega_j^{\alpha_{i_{k-1}}} \right|_{TV;[0,1]} \cdots \left| \omega_j^{\alpha_{i_0}} \right|_{TV;[0,1]} \left\| V_{\alpha_{i_0}} \cdots V_{\alpha_{i_k}} f \right\|_{\infty}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_T} (|R_m(T, x, f)|) &\leq \sum_{j=1}^n \lambda_j \sum_{\substack{\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m) \\ (\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_k}) \notin \mathcal{A}(m)}} \sqrt{T}^{\|\beta\|} \left| \omega_j^{\alpha_{i_k}} \right|_{TV;[0,1]} \left| \omega_j^{\alpha_{i_{k-1}}} \right|_{TV;[0,1]} \\
&\quad \cdots \left| \omega_j^{\alpha_{i_0}} \right|_{TV;[0,1]} \left\| V_{\alpha_{i_0}} \cdots V_{\alpha_{i_k}} f \right\|_{\infty}.
\end{aligned}$$

$\beta \in \mathcal{A}_{m+2}$  and  $\beta \notin \mathcal{A}_m$ , means that  $\|\beta\| = m+1$  or  $\|\beta\| = m+2$ . So for  $t > 1$ ,  $\sqrt{t}^{\|\beta\|} \leq \sqrt{t}^{m+2}$  and for  $0 < t \leq 1$ ,  $\sqrt{t}^{\|\beta\|} \leq \sqrt{t}^{m+1}$ . Therefore, by setting

$$C_{d,m,\mathbb{Q}_1} = \sum_{j=1}^n \lambda_j \sum_{\substack{\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k}) \in \mathcal{A}(m) \\ (\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_k}) \notin \mathcal{A}(m)}} \left| \omega_j^{\alpha_{i_k}} \right|_{TV;[0,1]} \left| \omega_j^{\alpha_{i_{k-1}}} \right|_{TV;[0,1]} \cdots \left| \omega_j^{\alpha_{i_0}} \right|_{TV;[0,1]}$$

we obtain

$$\mathbb{E}_{\mathbb{Q}_T} (|R_m(T, x, f)|) \leq C_{d,m,\mathbb{Q}_1} \sqrt{T}^{\|\beta\|} \left\| V_{\alpha_{i_0}} \cdots V_{\alpha_{i_k}} f \right\|_{\infty}$$

which is the announced upper bound.  $\square$

We now construct an approximation of  $\mathbb{E}(f(X_T^x))$  using the algorithm presented in Section 4.3. We will distinguish the case where  $f$  is smooth and where  $f$  is only Lipschitz.

#### 4.3.1 Approximation of $\mathbb{E}(f(X_T^x))$ when $f$ is Smooth

Assume that the paths  $\omega_{T,1}, \dots, \omega_{T,n}$  and the positive weights  $\lambda_1, \dots, \lambda_n$  define a cubature formula on Wiener space of degree  $m$  at time  $T$ , and let  $\mathbb{Q}_T$  be the probability measure with finite support defined by

$$\mathbb{Q}_T = \sum_{i=1}^n \lambda_i \delta_{\omega_{T,i}},$$

this is equivalent to  $\mathbb{E}_{\mathbb{Q}_T}(f(X_T^x)) = \sum_{i=1}^n \lambda_i f(X_T^x(\omega_{T,i}))$ . Recall that for  $\omega \in C_{0,bv}^0([0, T], \mathbb{R}^{d+1})$ ,  $\Phi_{T,x}(\omega) = X_T^x(\omega)$  defined by (4.1.5) is the solution at time  $T$  of the ordinary differential equation

$$dy_{t,x} = \sum_{i=0}^d V_i(y_{t,x}) d\omega^i(t) \quad (4.3.5)$$

with initial condition  $y_{0,x} = x \in \mathbb{R}^N$ . For a given smooth function  $f$ , we have

**Proposition 4.3.4.**

$$\begin{aligned} \sup_{x \in \mathbb{R}^N} \left| \mathbb{E}(f(X_T^x)) - \sum_{i=1}^n \lambda_i f(\Phi_{T,x}(\omega_{T,i})) \right| \\ \leq c \sqrt{T}^{m+1} \sup_{\alpha \in \mathcal{A}(m+2) \setminus \mathcal{A}(m)} \left\| V_{\alpha_{i_1}} \cdots V_{\alpha_{i_k}} f \right\|_{\infty}, \end{aligned} \quad (4.3.6)$$

where  $c$  is a constant independent of  $T$  and  $\alpha = (\alpha_{i_1}, \dots, \alpha_{i_k})$ .

*Proof.* From the definition of  $\Phi_{T,x}$ , we have

$$\sum_{i=1}^n \lambda_i f(\Phi_{T,x}(\omega_{T,i})) = \sum_{i=1}^n \lambda_i f(X_T^x(\omega_{T,i})) = \mathbb{E}_{\mathbb{Q}_T}(f(X_T^x)).$$

Therefore,

$$\begin{aligned} \mathbb{E}(f(X_T^x)) - \sum_{i=1}^n \lambda_i f(\Phi_{T,x}(\omega_{T,i})) &= \mathbb{E}(f(X_T^x)) - \mathbb{E}_{\mathbb{Q}_T}(f(X_T^x)) \\ &= (\mathbb{E} - \mathbb{E}_{\mathbb{Q}_T})(f(X_T^x)). \end{aligned}$$

Using the fact that the stochastic Taylor expansion of  $f$  is given by

$$f(X_T^x) = \sum_{\alpha \in \mathcal{A}(m)} V_{\alpha_{i_1}} \cdots V_{\alpha_{i_k}} f(x) \mathbb{E}(B_T^{\circ\alpha}) + R_m(T, x, f),$$

where  $B_T^{\circ\alpha} = \int_{0 < t_1 < \cdots < t_k < T} \circ dB_{t_1}^{\alpha_{i_1}} \circ \cdots \circ dB_{t_k}^{\alpha_{i_k}}$ , the previous expression becomes

$$\begin{aligned} (\mathbb{E} - \mathbb{E}_{\mathbb{Q}_T})(f(X_T^x)) &= (\mathbb{E} - \mathbb{E}_{\mathbb{Q}_T})(R_m(T, x, f)) \\ &+ (\mathbb{E} - \mathbb{E}_{\mathbb{Q}_T}) \left( \sum_{\alpha \in \mathcal{A}(m)} V_{\alpha_{i_1}} \cdots V_{\alpha_{i_k}} f(x) \mathbb{E}(B_T^{\circ\alpha}) \right). \end{aligned}$$

This implies that,

$$\begin{aligned} |(\mathbb{E} - \mathbb{E}_{\mathbb{Q}_T})(f(X_T^x))| &\leq |\mathbb{E}(R_m(T, x, f)) - \mathbb{E}_{\mathbb{Q}_T}(R_m(T, x, f))| \\ &+ \left| \sum_{\alpha \in \mathcal{A}(m)} V_{\alpha_{i_1}} \cdots V_{\alpha_{i_k}} f(x) (\mathbb{E} - \mathbb{E}_{\mathbb{Q}_T})(B_T^{\circ\alpha}) \right|. \end{aligned}$$

Using Lemma 4.3.1, we have  $\mathbb{E}(B_T^{\circ\alpha}) = \mathbb{E}_{\mathbb{Q}_T}(B_T^{\circ\alpha})$  and it follows

$$\begin{aligned} |(\mathbb{E} - \mathbb{E}_{\mathbb{Q}_T})(f(X_T^x))| &\leq |\mathbb{E}(R_m(T, x, f))| + |\mathbb{E}_{\mathbb{Q}_T}(R_m(T, x, f))| \\ &\leq \mathbb{E} \left( |R_m(T, x, f)| \right) + \mathbb{E}_{\mathbb{Q}_T} \left( |R_m(T, x, f)| \right) \\ &\leq \sqrt{\mathbb{E} \left( R_m(T, x, f)^2 \right)} + \mathbb{E}_{\mathbb{Q}_T} \left( |R_m(T, x, f)| \right). \end{aligned}$$

Using Proposition 2.2.9 and Lemma 4.3.2, we conclude that

$$\begin{aligned} \sup_{x \in \mathbb{R}^N} |(\mathbb{E} - \mathbb{E}_{\mathbb{Q}_T})(f(X_T^x))| &\leq c_1 \sqrt{T}^{m+1} \sup_{\alpha \in \mathcal{A}(m+2) \setminus \mathcal{A}(m)} \|V_{\alpha_{i_1}} \cdots V_{\alpha_{i_k}} f\|_{\infty} \\ &+ c_2 \sqrt{T}^{m+1} \sup_{\alpha \in \mathcal{A}(m+2) \setminus \mathcal{A}(m)} \|V_{\alpha_{i_1}} \cdots V_{\alpha_{i_k}} f\|_{\infty}, \end{aligned}$$

this is equivalent to

$$\sup_{x \in \mathbb{R}^N} \left| \mathbb{E}(f(X_T^x)) - \mathbb{E}_{\mathbb{Q}_T}(f(X_T^x)) \right| \leq c\sqrt{T}^{m+1} \sup_{\alpha \in \mathcal{A}(m+2) \setminus \mathcal{A}(m)} \left\| V_{\alpha_{i_1}} \cdots V_{\alpha_{i_k}} f \right\|_{\infty}.$$

□

We observe from Proposition 4.3.4 that the quantity  $\mathbb{E}(f(X_T^x))$  is approximated by a weighted sum of solutions of ODEs,  $\Phi_{T,x}(\omega)$ . Therefore the task of solving an SDE is replaced by the one of solving  $N$  ODEs. In order for  $\mathbb{E}_{\mathbb{Q}_T}(f(X_T^x))$  to represent a good approximation of  $\mathbb{E}(f(X_T^x))$ , the upper bound in Equation (4.3.6) needs to be as small as possible. This may be done by dividing the interval  $[0, T]$  into  $k$  small subintervals, and then we consecutively apply the approximation to these intervals in the following way: Consider  $0 = t_0 < t_1 < \cdots < t_k = T$  and  $s_\ell = t_\ell - t_{\ell-1}$  for  $\ell = 1, \dots, k$ . Define a sequence of random variables  $(Y_\ell)_{0 \leq \ell \leq k}$  by

$$\mathbb{P}\left(Y_{j+1} = \Phi_{s_{j+1}, x}\left(\omega_{s_{j+1}, i}\right) \mid Y_j = x\right) = \lambda_i \quad (4.3.7)$$

for  $j = 0, \dots, k-1$  and  $i = 1, \dots, n$ . So, we obtain  $Y_{j+1}$  by following (with probability  $\lambda_i$ ) the solution of the ODE (4.3.5) driven by  $\omega_{s_{j+1}, i}$  starting at  $Y_j$  until time  $s_{j+1}$ . Notice that  $(Y_\ell)_{0 \leq \ell \leq k}$  satisfies the Markov property, i.e. the distribution of  $Y_{\ell+1}$  given  $Y_\ell$  is independent of  $Y_0, \dots, Y_{\ell-1}$ . We then have the following theorem:

**Theorem 4.3.5.** (*Lyons and Victoir, 2004*). *Consider the Markov chain  $(Y_\ell)_{0 \leq \ell \leq k}$  defined in Equation (4.3.7). We have*

$$\begin{aligned} \sup_{x \in \mathbb{R}^N} |\mathbb{E}(f(Y_k) \mid Y_0 = x) - \mathbb{E}(f(X_T^x))| \\ \leq c \sum_{j=1}^k s_j^{(m+1)/2} \sup_{\alpha \in \mathcal{A}(m+2) \setminus \mathcal{A}(m)} \left\| V_{\alpha_{i_1}} \cdots V_{\alpha_{i_k}} P_{T-t_j} f \right\|_{\infty} \end{aligned} \quad (4.3.8)$$

for some constant  $c$  and where  $P_t f(x) = \mathbb{E}(f(X_t^x))$ .

Considering the above notations, we define the path  $\omega_{s_1, i_1} \otimes \cdots \otimes \omega_{s_k, i_k}(t)$  for  $0 \leq \ell \leq k-1$  and  $t_\ell \leq t \leq t_{\ell+1}$  as

$$\omega_{s_1, i_1} \otimes \cdots \otimes \omega_{s_k, i_k}(t) := \omega_{s_1, i_1} + \cdots + \omega_{s_\ell, i_\ell} + \omega_{s_{\ell+1}, i_{\ell+1}}(t - t_\ell)$$

where  $s_i = t_i - t_{i-1}$ , and  $i_1, \dots, i_{\ell+1} \in \{0, 1, \dots, k\}$ . Notice that  $\omega_{s_1, i_1} \otimes \cdots \otimes \omega_{s_k, i_k} \in C_{0, bv}^0([0, T], \mathbb{R}^{d+1})$  is the path obtained by, consecutively, following the paths  $\omega_{s_1, i_1}, \dots, \omega_{s_k, i_k}$ . Here  $\omega_{s_2, i_2}$  is understood to be attached to the end point of the path  $\omega_{s_1, i_1}$  and so on. Then, given the initial condition



$Y_0 = x$ , the random variable  $Y_k$  takes value  $\Phi_{T,x}(\omega_{s_1,i_1} \otimes \cdots \otimes \omega_{s_k,i_k})$  with probability  $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$ . Therefore,  $\mathbb{E}(f(X_T^x))$  can be approximated by

$$\begin{aligned} \mathbb{E}(f(Y_k) | Y_0 = x) &= \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} f\left(\Phi_{T,x}(\omega_{s_1,i_1} \otimes \cdots \otimes \omega_{s_k,i_k})\right) \\ &= \mathbb{E}_{\mathbb{Q}_T^k}(f(X_T^x)) \end{aligned}$$

where

$$\mathbb{Q}_T^k = \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \delta_{\omega_{s_1,i_1} \otimes \cdots \otimes \omega_{s_k,i_k}}.$$

One can see this algorithm as a  $n$ -ary tree. At each node, we save a weight and an element of  $\mathbb{R}^N$ . We begin at time 0 from the root node  $(1, x)$ . In the first step, we construct at time  $s_1$  the first  $n$  nodes by computing the  $n$  solutions  $(\Phi_{s_1,x}(\omega_{s_1,i}))_{1 \leq i \leq n}$  of the ODEs (4.3.5) corresponding to the paths  $\omega_{s_1,1}, \dots, \omega_{s_1,n}$  with initial value  $x$  and with probability  $\lambda_1, \dots, \lambda_n$ , respectively. Thus, we save

$$\{(v_1^1, f_1^1), \dots, (v_n^1, f_n^1)\} = \{(\lambda_1, \Phi_{s_1,x}(\omega_{s_1,1})), \dots, (\lambda_n, \Phi_{s_1,x}(\omega_{s_1,n}))\}.$$

Then, for each  $1 \leq j \leq n$ , we compute the  $n$  solutions at time  $s_2$  of the ODEs corresponding to the paths  $\omega_{s_2,1}, \dots, \omega_{s_2,n}$  with initial value  $\Phi_{s_1,x}(\omega_{s_1,j})$  and for each  $j$ , we save  $(v_1^{2,j}, f_1^{2,j}), \dots, (v_n^{2,j}, f_n^{2,j})$  where  $v_i^{2,j} = v_j^1 \lambda_i$  and  $f_i^{2,j} = \Phi_{s_2,f_j^1}(\omega_{s_2,i})$ . This procedure is iterated  $k-2$  times. The last stage is obtained as follows: From each node  $j$ ,  $1 \leq j \leq n^{k-1}$  of the stage  $k-1$ , we obtain  $(v_1^{k,j}, f_1^{k,j}), \dots, (v_n^{k,j}, f_n^{k,j})$  where  $(v_i^{k,j}, f_i^{k,j})$  are computed as mentioned previously and the sum

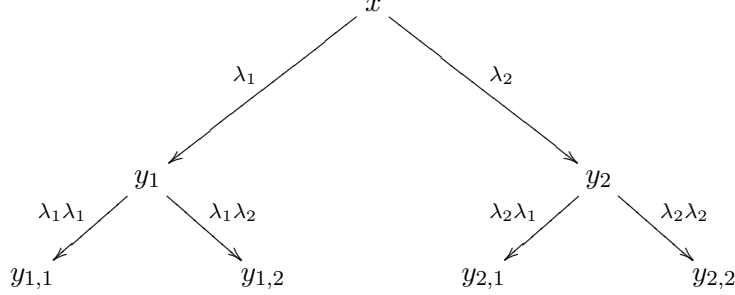
$$\sum_j \sum_{i=1}^n v_i^{k,j} f_i^{k,j}$$

is our approximation for  $\mathbb{E}(f(X_T^x))$ .

**Example 4.3.6.** considering  $n = 2$  and  $k = 2$ , we construct a cubature formula with two paths by discretizing the interval  $[0, T]$  into 2 small intervals. Let  $[0, T] = [0, s_1] \cup [s_1, s_1 + s_2]$  and

$$\begin{aligned} y_i &= \Phi_{s_1,x}(\omega_{s_1,i}) \\ y_{i,j} &= \Phi_{s_2,\Phi_{s_1,x}(\omega_{s_1,i})}(\omega_{s_2,j}) \end{aligned}$$

where  $\Phi_{t,x}(\omega)$  is defined as previously. We then have the following diagram:



Then

$$\mathbb{E}_{\mathbb{Q}_T^2}(f(X_T^x)) = \lambda_1 \lambda_1 f(y_{1,1}) + \lambda_1 \lambda_2 f(y_{1,2}) + \lambda_2 \lambda_1 f(y_{2,1}) + \lambda_2 \lambda_2 f(y_{2,2})$$

#### 4.4 Approximation of $\mathbb{E}(f(X_T^x))$ when $f$ is Lipschitz

In the case when  $f$  is only Lipschitz, we have the following regularity result due to Kusuoka and Strook, 1987 and Kusuoka (2001). The proof is based on Malliavin calculus.

**Theorem 4.4.1.** *Let  $f$  be a Lipschitz function,  $\alpha_1, \dots, \alpha_k \in \mathcal{A}_1$  be  $k$  multi-indices and  $V_{[\alpha_1]}, \dots, V_{[\alpha_k]}$  be a family of vector fields satisfying the UFG condition (1), that is, for each  $\alpha_i, i = 1, \dots, k$  there exists an integer  $l_i$  and a function  $\varphi_{\alpha_i, \beta} \in C_b^\infty(\mathbb{R}^N)$  where  $\beta \in \mathcal{A}_1(l_i)$  such that*

$$V_{[\alpha_i]} = \sum_{\beta \in \mathcal{A}_1(l_i)} \varphi_{\alpha_i, \beta} V_{[\beta]}.$$

Then, there exists a constant  $C$  such that

$$\left\| V_{[\alpha_1]} \cdots V_{[\alpha_k]} P_t f \right\|_\infty \leq \frac{C t^{1/2}}{t^{\|\alpha_1 * \dots * \alpha_k\|/2}} \|\nabla f\|_\infty. \quad (4.4.1)$$

This theorem states, loosely speaking, that  $P_t f$  is smooth even when  $f$  is not (see also Theorem 6.2 in Litterer and Lyons, 2006).

**Corollary 4.4.2.** *Under assumptions of Theorem 4.4.1, let  $0 = t_0 < t_1 < \dots < t_k = T$  and  $s_\ell = t_\ell - t_{\ell-1}$ . Also, consider the Markov chain  $(Y_i)_{0 \leq i \leq k}$  defined in Equation (4.3.7). We have the following estimate:*

$$\begin{aligned} \sup_{x \in \mathbb{R}^N} |\mathbb{E}(f(Y_k) | Y_0 = x) - \mathbb{E}(f(X_T^x))| \\ \leq C \|\nabla f\|_\infty \left( s_k^{1/2} + \sum_{i=1}^{k-1} \frac{s_i^{(m+1)/2}}{(T - t_i)^{m/2}} \right) \end{aligned} \quad (4.4.2)$$

This corollary indicates that the algorithm described in the previous section remains valid when  $f$  is considered to be only Lipschitz continuous. It was pointed out in Litterer and Lyons, 2006 that the estimates in the above proposition require  $V_0$  to have a formal degree at most 2 and that if  $V_0$  has a higher formal degree, the estimate in Theorem 4.4.1 will change and the bound in Corollary 4.4.2 will change accordingly. The proof of the Corollary 4.4.2 is based on Proposition 4.3.4 and may be found in Lyons and Victoir, 2004.

The cubature formula on Wiener space can be viewed as a restriction of the more general framework presented in Chapter 3. In fact, the concept of cubature formula on Wiener space is similar to the concept of  $m$ -moment similar family presented in Chapter 3 in the following sense:

Let  $(\omega_i, \lambda_i)_{i=1, \dots, n}$  be some paths and weights defining a cubature formula on Wiener space of degree  $m$ . Define the random variable  $\eta$  by:

$$P(\eta = B_T^{\circ\alpha}(\omega_i)) = \lambda_i \quad \text{for } i = 1, \dots, n.$$

Then, the family of random variables  $\{Z_\alpha; \alpha \in \mathcal{A}_0\}$  defined as

$$Z_{(\alpha_0)} = 1, \quad Z_\alpha = \eta, \quad \text{if } \|\alpha\| \leq m \quad \text{and} \quad Z_\alpha = 0 \quad \text{otherwise}$$

is an  $m$ -moment similar family.

**Remark 4.4.3.** *It is not difficult to verify that  $\mathbb{E}[|Z_\alpha|^k] < \infty$  for all  $\alpha \in \mathcal{A}_0$  and  $k \geq 1$ . For  $k = 1$ , and by considering the fact that*

$$\mathbb{E}(B_T^{\circ\alpha}) = \sum_{i=1}^n \lambda_i B_T^{\circ\alpha}(\omega_i), \quad (4.4.3)$$

*we obtain  $\mathbb{E}[Z_\alpha] = \mathbb{E}(B_T^{\circ\alpha})$ . It is proved in Bayer (March 2008) that for all  $k$  verifying  $\|\alpha_1\| + \dots + \|\alpha_k\| \leq m$ , we have*

$$\mathbb{E}[Z_{\alpha_1} \dots Z_{\alpha_k}] = \mathbb{E}(B_T^{\circ\alpha_1} \dots B_T^{\circ\alpha_k}).$$

Although the class of moment similar families of random variables is larger than the class of cubature formulas on Wiener space, it is not easier to construct them. Thus, it is also important to consider the class of cubature formulas on Wiener space.

## 4.5 Examples of Constructions of Cubature on Wiener Space

As mentioned by Bayer (March 2008), the actual construction of cubature paths is a difficult task. In this Section, we only present a few examples of (piecewise linear) cubature paths on Wiener space of degree 3 and 5 obtained by Lyons and Victoir, 2004.

### 4.5.1 Example of Cubature Formula of Degree 3 on Wiener Space

We consider some given points  $x_1, \dots, x_n \in \mathbb{R}^d$  and positive weights  $\lambda_1, \dots, \lambda_n$  which are supposed to define a classical cubature formula of degree 3 with respect to the standard  $d$ -dimensional Gaussian measure. The most classical example is given by  $(x_i, \lambda_i)_{i=1, \dots, n}$  where  $n = 2^d$ ,  $x_i$  are  $2^d$  points  $\{-1, +1\}^d$  and  $\lambda_i = 2^{-d}$  (see Victoir (2004) for details).

**Remark 4.5.1.** *The size of this cubature formula is not optimal since by Tchakaloff's theorem, one can construct a cubature formula of degree 3 with the number of points  $n \leq \frac{d(d+1)(d+2)}{6}$ .*

The cubature on Wiener space of degree 3 for a  $d$ -dimensional Brownian motion is then given by  $(\omega_i, \lambda_i)_{i=1, \dots, n}$  where  $\omega_i(t) = t(1, x_i^1, \dots, x_i^d)$  for  $i = 1, \dots, n$ .

### 4.5.2 Example of Cubature Formula of Degree 5 on Wiener Space

Let  $x_1, \dots, x_n \in \mathbb{R}^d$  and  $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$  be some points and weights defining a cubature formula of degree 5 with respect to a  $d$ -dimensional Gaussian measure.

**Example 4.5.2.** *Consider  $d = 1$ , and fix  $n = 3$ . The points  $(x_1, x_2, x_3) = (-\sqrt{3}, 0, \sqrt{3})$ , and weights  $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$  define a classical cubature formula of degree 5.*

The construction of cubature formula of degree 5 on Wiener space is given by the following theorem:

**Theorem 4.5.3** (Lyons and Victoir, 2004). *Let  $d \in \mathbb{N}^*$ . Let the points  $x_1, \dots, x_n \in \mathbb{R}^d$  and the positive weights  $\lambda_1, \dots, \lambda_n$  define a cubature formula of degree 5. Let  $\omega = (\omega^1, \dots, \omega^d) \in C_{0,bv}^0([0, 1], \mathbb{R}^d)$  be such that*

$$\left\{ \begin{array}{ll} \forall i \in \{1, \dots, d\}, & \int_0^1 \omega^i(t) dt = \int_0^1 \omega^i(t)^2 dt = \frac{1}{2}, \\ \forall i \in \{1, \dots, d\}, & \omega^i(0) = 0 \text{ and } \omega^i(1) = 1, \\ \text{for } 1 \leq i < j \leq d, & \int_0^1 \omega^i(t) d\omega^j(t) - \int_0^1 \omega^j(t) d\omega^i(t) = 1, \\ \text{for } 1 \leq i < j \leq d, & \int_0^1 \omega^i(t)^2 d\omega^j(t) + \int_0^1 \omega^j(t)^2 d\omega^i(t) = 1. \end{array} \right. \quad (4.5.1)$$

Then, for  $i = 1, \dots, n$ , the paths defined by

$$\begin{aligned} \omega_i(t) &= \left( t, x_i^1 \omega^1(t), \dots, x_i^d \omega^d(t) \right), \\ \omega_{n+i}(t) &= \left( t, x_i^1 \omega^d(t), \dots, x_i^d \omega^1(t) \right), \end{aligned}$$

and the positive weights  $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{2n}$  given by  $\tilde{\lambda}_i = \lambda_i/2 = \tilde{\lambda}_{n+i}$  define a cubature formula on Wiener space of degree 5.

Lyons and Victoir, 2004 constructed piecewise linear solutions for Equations (4.5.1). For example, for a one dimension Brownian motion (i.e  $d = 1$ ), they obtained that the paths  $\left(t, -\sqrt{3}\omega_1^{[1]}(t)\right), (t, 0), \left(t, \sqrt{3}\omega_1^{[1]}(t)\right)$  and the positive weights  $\frac{1}{6}, \frac{2}{3}, \frac{1}{6}$  define a quadrature formula on Wiener space of degree 5, where

$$\begin{aligned} \omega_1^{[1]} : [0, 1] &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} \frac{1}{2}(4 - \sqrt{22})t, & 0 \leq t \leq \frac{1}{3}, \\ \frac{1}{6}(4 - \sqrt{22}) + (-1 + \sqrt{22})\left(t - \frac{1}{3}\right), & \frac{1}{3} \leq t \leq \frac{2}{3}, \\ \frac{1}{6}(2 + \sqrt{22}) + \frac{1}{2}(4 - \sqrt{22})\left(t - \frac{2}{3}\right), & \frac{2}{3} \leq t \leq 1. \end{cases} \end{aligned}$$

## Chapter 5

# Applications of Kusuoka and Lyons-Victoir Approximation Schemes

As mentionned in Chapter 3, the higher-order scheme introduced by Kusuoka greatly improves the speed and the accuracy of the numerical weak approximation of SDEs. In this Chapter, we present high-order schemes developed by Ninomiya and Victoir, 2008 and Ninomiya and Ninomiya, 2009. An intuitive explanation of these schemes is as follows: Consider the semi-group of linear operators  $P_t$  defined in (1.0.10) by

$$(P_t f)(x) = \mathbb{E}[f(X_t^x)], \quad \text{where } t \in [0, T], f \in C_b^\infty(\mathbb{R}^N) \quad \text{and} \\ X(t, x) = X_t(x) = X_t^x = x + \sum_{j=0}^d \int_0^t V_j(X_s^x) \circ dB^j(s) \quad . \quad (5.0.1)$$

In order to compute  $\mathbb{E}[f(X_T^x)]$ , we want to construct a linear operator  $Q_t$  such that  $Q_t$  approximates  $P_t$  in the sense that  $(P_t - Q_t)f(x) = \mathcal{O}(t^{m+1})$  for small  $t$ , this gives us an  $m$ -th order approximation of  $P_t$ . The key idea here is to construct a Markov process  $(\bar{X}_t(x))_{t \in [0, T]}$  starting at  $x$  such that  $Q_t f(x) = \mathbb{E}(f(\bar{X}_t(x)))$  and for any partition  $0 = t_0 < t_1 < \dots < t_n = T$  of  $[0, T]$ ,

$$Q_T f(x) = \mathbb{E}[f(\bar{X}_{s_n} \circ \dots \circ \bar{X}_{s_1}(x))]$$

where  $s_k = t_k - t_{k-1}$ , for  $k = 1, \dots, n$  and  $\bar{X}_{s_k} \circ \bar{X}_{s_{k-1}}(x) = \bar{X}_{s_k}(\bar{X}_{s_{k-1}}(x))$ . An algebraic structure of this idea is taken from Tanaka (2008) as follows: Consider the solution  $u(t, x)$  to the heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) &= Lu(t, x) \\ u(0, x) &= f(x) \end{cases}, \quad (5.0.2)$$

where

$$Lf = V_0 f + \frac{1}{2} \sum_{i=1}^d V_i^2 f.$$

$$u(t, x) = \mathbb{E}[f(X_t^x)] = P_t f(x).$$

Briefly speaking, for small  $t$ ,  $u(t, x) = (e^{tL} f)(x)$ . Thus, the operator

$$P_t = e^{tL} = \sum_{j=0}^m \frac{t^j}{j!} L^j + \mathcal{O}(t^{m+1}).$$

For  $L_0 f(x) = V_0 f(x)$  and  $L_i f(x) = \frac{1}{2} V_i^2 f(x)$  for  $i = 1, \dots, d$ , we have

$$e^{tL_i} = \sum_{j=0}^m \frac{t^j}{j!} L_i^j + \mathcal{O}(t^{m+1}).$$

The operator  $Q_t$  is then constructed as a combination of  $L_i$  through

$$Q_t = \sum_{j=1}^k \xi_j e^{t_{1,j} L_{1,j}} \dots e^{t_{\ell_j,j} L_{\ell_j,j}}$$

such that

$$e^{tL} - \sum_{j=1}^k \xi_j e^{t_{1,j} L_{1,j}} \dots e^{t_{\ell_j,j} L_{\ell_j,j}} = \mathcal{O}(t^{m+1}) \quad (5.0.3)$$

where  $t_{i,j} > 0$ ,  $L_{i,j} \in \{L_0, L_1, \dots, L_d\}$  and  $\xi_j$  are points in  $[0, 1]$  with  $\sum_{j=1}^k \xi_j = 1$ . We finally obtain

$$\begin{aligned} P_t f = e^{tL} f &\simeq \sum_{j=1}^k \xi_j e^{t_{1,j} L_{1,j}} \dots e^{t_{\ell_j,j} L_{\ell_j,j}} f \\ &\simeq \sum_{j=1}^k \xi_j \mathbb{E} \left[ f \left( \bar{X}_1(t_{1,j}, \bar{X}_2(t_{1,j}, \dots \bar{X}_{\ell}(t_{\ell_j,j}, \cdot) \dots)) \right) \right]. \end{aligned}$$

For examples, we have the following:

**Example 5.0.4.** • for  $d = 1$ , we have

$$\begin{aligned} e^{tL} &= I + tL + \frac{t^2}{2} L^2 + \mathcal{O}(t^3) \\ e^{tL_0} e^{tL_1} &= \left( I + tL_0 + \frac{t^2}{2} L_0^2 + \mathcal{O}(t^3) \right) \left( I + tL_1 + \frac{t^2}{2} L_1^2 + \mathcal{O}(t^3) \right) \\ &= I + tL + \frac{t^2}{2} (L_0^2 + L_1^2 + 2L_0 L_1) + \mathcal{O}(t^3). \end{aligned}$$

Then

$$\begin{aligned} e^{tL} - e^{tL_0} e^{tL_1} &= \mathcal{O}(t^2) \\ e^{tL} - \frac{1}{2} e^{tL_0} e^{tL_1} - \frac{1}{2} e^{tL_1} e^{tL_0} &= \mathcal{O}(t^3). \end{aligned}$$

Finally, one needs to obtain a stochastic representation for  $\frac{1}{2} e^{tL_0} e^{tL_1} + \frac{1}{2} e^{tL_1} e^{tL_0}$ .

- For  $d = 2$ , we have

$$\begin{aligned} e^{tL} &= I + tL + \frac{t^2}{2} L^2 + \mathcal{O}(t^3) \\ e^{\frac{t}{2}L_0} e^{tL_1} &= \left( I + \frac{t}{2} L_0 + \frac{t^2}{8} L_0^2 + \mathcal{O}(t^3) \right) \left( I + tL_1 + \frac{t^2}{2} L_1^2 + \mathcal{O}(t^3) \right) \\ &= I + t \left( \frac{1}{2} L_0 + L_1 \right) + \frac{t^2}{2} \left( \frac{1}{4} L_0^2 + L_1^2 + L_0 L_1 \right) + \mathcal{O}(t^3) \\ e^{tL_2} e^{\frac{t}{2}L_0} &= \left( I + tL_2 + \frac{t^2}{2} L_2^2 + \mathcal{O}(t^3) \right) \left( I + \frac{t}{2} L_0 + \frac{t^2}{8} L_0^2 + \mathcal{O}(t^3) \right) \\ &= I + t \left( \frac{1}{2} L_0 + L_2 \right) + \frac{t^2}{2} \left( \frac{1}{4} L_0^2 + L_2^2 + L_2 L_0 \right) + \mathcal{O}(t^3). \end{aligned}$$

Let  $A = e^{\frac{t}{2}L_0} e^{tL_1} e^{tL_2} e^{\frac{t}{2}L_0}$ . We obtain,

$$A = I + tL + \frac{t^2}{2} (L_0^2 + L_1^2 + L_2^2 + L_2 L_0 + 2L_1 L_2 + L_1 L_0 + L_0 L_2 + L_0 L_1) + \mathcal{O}(t^3).$$

which implies that

$$e^{tL} - e^{\frac{t}{2}L_0} e^{tL_1} e^{tL_2} e^{\frac{t}{2}L_0} = \mathcal{O}(t^2), \quad (5.0.4)$$

$$e^{tL} - \frac{1}{2} e^{\frac{t}{2}L_0} e^{tL_1} e^{tL_2} e^{\frac{t}{2}L_0} - \frac{1}{2} e^{\frac{t}{2}L_0} e^{tL_2} e^{tL_1} e^{\frac{t}{2}L_0} = \mathcal{O}(t^3) \quad (5.0.5)$$

and one will now have to find a stochastic representation for  $\frac{1}{2} e^{\frac{t}{2}L_0} e^{tL_1} e^{tL_2} e^{\frac{t}{2}L_0} + \frac{1}{2} e^{\frac{t}{2}L_0} e^{tL_2} e^{tL_1} e^{\frac{t}{2}L_0}$  in order to obtain a second-order approximation.

The Ninomiya-Victoir and Ninomiya-Ninomiya schemes can be viewed in a non-trivial way as particular cases of Lyons-Victoir's methodology and thus of Kusuoka. The idea is to transform the problem of solving an SDE into a problem of solving ODEs. The two schemes are quite similar but the algorithms themselves differ significantly. This Chapter is organised as follows: In Section 5.1, we develop the Ninomiya-Victoir scheme. The algorithm of this approximation scheme is then applied to solve an option pricing problem and then to solve an optimal portfolio strategies problem. Numerical simulations are performed at the end of this section and results obtained using the



new approximation scheme are compared with the traditional Euler-Maruyama scheme.

The Ninomiya–Ninomiya approximation scheme is presented in Section 5.2. The algorithm is utilized to solve a problem of pricing Asian options under the Heston stochastic volatility model.

## 5.1 Ninomiya–Victoir Scheme

We consider the stochastic differential equation (5.0.1). We want to compute  $\mathbb{E}(f(X(T, x)))$  where  $f$  is a given function and  $T > 0$ . The purpose of this Section is to present the numerical scheme due to Ninomiya and Victoir, 2008. The construction of the scheme is based on Theorem 5.1.1.

### 5.1.1 Presentation of the New Algorithm

**Theorem 5.1.1.** *Let  $(\Lambda_i, \zeta_i)_{i \in \{1, \dots, n\}}$  be  $n$  independent random variables, where each  $\Lambda_i$  denotes a Bernoulli random variable and  $\zeta_i$  a standard  $d$ -dimensional normal random variable. Each  $\Lambda_i$  is independent of  $\zeta_i$ ,  $i \in \{1, \dots, n\}$ . Consider the family  $\{\bar{X}_{(k)}\}_{k=0, \dots, n}$  of random variables defined as*

$$\begin{aligned} \bar{X}_{(0)} &= x, \\ \bar{X}_{(k+1)} &= \exp\left(\frac{T}{2n}V_0\right) \exp\left(\sqrt{\frac{T}{n}}\zeta_k^1 V_1\right) \cdots \exp\left(\sqrt{\frac{T}{n}}\zeta_k^d V_d\right) \exp\left(\frac{T}{2n}V_0\right) \bar{X}_k \\ &\quad \text{if } \Lambda_k = +1, \\ \text{and} \\ \bar{X}_{(k+1)} &= \exp\left(\frac{T}{2n}V_0\right) \exp\left(\sqrt{\frac{T}{n}}\zeta_k^d V_d\right) \cdots \exp\left(\sqrt{\frac{T}{n}}\zeta_k^1 V_1\right) \exp\left(\frac{T}{2n}V_0\right) \bar{X}_k \\ &\quad \text{if } \Lambda_k = -1. \end{aligned}$$

Then, for an arbitrary Lipschitz continuous function  $f$ ,

$$|E[f(\bar{X}_{(n)})] - E[f(X(T, x))]| \leq \frac{cf}{n^2}, \quad (5.1.1)$$

that is, the new algorithm is of order 2.

Considering the fact that for a smooth vector field  $V$ ,  $\exp(tV)(x)$  is the solution at time  $t$  of the ODE

$$\begin{aligned} \frac{dY}{dt}(t, x) &= V(Y(t, x)), \\ Y(0, x) &= x, \end{aligned}$$

at each stage  $k$ ,  $1 \leq k \leq n$ , one will have to solve  $d + 2$  ODEs in order to compute  $\exp\left(\frac{T}{2n}V_0\right) \exp\left(\sqrt{\frac{T}{n}}\zeta_k^d V_d\right) \cdots \exp\left(\sqrt{\frac{T}{n}}\zeta_k^1 V_1\right) \exp\left(\frac{T}{2n}V_0\right) \bar{X}_k$ . First, along the vector field  $V_0$  from  $t = 0$  to  $t = \frac{T}{2n}$  with starting point  $\bar{X}_k$ , then along  $V_d$  from  $t = 0$  to  $t = \sqrt{\frac{T}{n}}\zeta_k^d$  with starting point the solution we have obtained from solving the previous ODE. We repeat similar operations  $d+2$  times.

**Remark 5.1.2.** *As mentioned by Ninomiya and Victoir, 2008, if there are no closed form solutions to these ODEs, one will need to use numerical methods in order to approximate the solutions. these methods have to be of order  $\mathcal{O}(s^3)$  and  $\mathcal{O}(s^6)$  for  $\exp(sV_0)$  and  $\exp(sV_i)$ , respectively.*

Once the random variable  $\bar{X}_{(n)}$  is constructed using the algorithm presented above, one can use Monte Carlo or Quasi-Monte Carlo methods in order to approximate  $E[f(\bar{X}_{(n)})]$ .

### 5.1.2 Implementation

We implement the Ninomiya-Victoir approximation scheme to two different financial problems. First, to an option pricing problem where we consider the pricing of an Asian call option (Ninomiya and Victoir, 2008). Then, to an optimal portfolio strategies problem where we consider a Stock-Bond-Cash allocation problem presented in Fukaya (2006).

#### 5.1.2.1 Option Pricing

We consider the price of an Asian call option of European type. We assume that the option is written on an asset whose price process follows the Heston stochastic volatility model. More precisely, we have

$$\begin{aligned} X_1(t, x) &= x_1 + \int_0^t \mu X_1(s, x) ds + \int_0^t X_1(s, x) \sqrt{X_2(s, x)} dB^1(s), \\ X_2(t, x) &= x_2 + \int_0^t \kappa(\theta - X_2(s, x)) ds \\ &\quad + \int_0^t \sigma \sqrt{X_2(s, x)} \left( \rho dB^1(s) + \sqrt{1 - \rho^2} dB^2(s) \right), \end{aligned} \quad (5.1.2)$$

where  $X_1$  describes the asset price movement and  $X_2$  is the diffusion process for the volatility. Here  $x = (x_1, x_2) \in (\mathbb{R}_+^*)^2$  is the initial value for price and volatility, respectively.  $(B^1(t), B^2(t))$  is a two-dimensional Brownian motion with correlation  $\rho$ ,  $(-1 < \rho < 1)$  and  $\kappa$ ,  $\theta$  and  $\mu$  are some positive coefficients such that

$$2\kappa\theta - \sigma^2 > 0.$$

This condition is to ensure the existence and uniqueness of a positive solution for Equation (5.1.2). The pay-off of an Asian call option on the above-mentioned asset at the maturity  $T$  with strike price  $K$  is  $\max\left(\frac{X_3(T,x)}{T} - K, 0\right)$  where  $(X_3(t,x))_t$  is a new process defined by:

$$X_3(t, x) = \int_0^t X_1(s, x) ds.$$

Therefore, the price of the option is  $D \times E \left[ \max\left(\frac{X_3(T,x)}{T} - K, 0\right) \right]$  where  $D$  is an appropriate discount factor.

We introduce the following notation in order to express the model in Itô form and then transform it into Stratonovich form.

$$\begin{aligned} B^0(t) &= t \\ X(t, x) &= (X_1(t, x), X_2(t, x), X_3(t, x))' \\ &= \begin{pmatrix} X_1(t, x) \\ X_2(t, x) \\ X_3(t, x) \end{pmatrix}. \end{aligned}$$

The Itô form of the model is then

$$X(t, x) = x + \sum_{i=0}^2 \int_0^t \tilde{V}_i(X(s, x)) dB^i(s), \quad (5.1.3)$$

where  $x = (x_1, x_2, 0)'$  and  $\tilde{V}_0, \tilde{V}_1, \tilde{V}_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  are defined as follows:

$$\tilde{V}_0(y) = \tilde{V}_0((y_1, y_2, y_3)') = (\mu y_1, \kappa(\theta - y_2), y_1)' \quad (5.1.4)$$

$$\tilde{V}_1(y) = \tilde{V}_1((y_1, y_2, y_3)') = (y_1 \sqrt{y_2}, \rho \sigma \sqrt{y_2}, 0)' \quad (5.1.5)$$

$$\tilde{V}_2(y) = \tilde{V}_2((y_1, y_2, y_3)') = (0, \beta \sqrt{1 - \rho^2} \sqrt{y_2}, 0)'. \quad (5.1.6)$$

The next step is to transform Equation (5.1.3) into Stratonovich form.

$$X(t, x) = x + \sum_{i=0}^2 \int_0^t V_i(X(s, x)) \circ dB^i(s), \quad (5.1.7)$$

where  $x$  is defined as previously. Using the formula equation (1.0.4) we have that

$$\begin{aligned} V_0^i &= \tilde{V}_0^i - \frac{1}{2} \sum_{j=1}^2 \sum_{k=1}^3 \tilde{V}_j^k \frac{\partial \tilde{V}_j^i}{\partial x_k}, \\ V_j^i &= \tilde{V}_j^i, \quad j = 1, 2. \end{aligned}$$

That is,

$$\begin{aligned}
 V_0^1(y) &= \mu y_1 - \frac{1}{2} \left( \sqrt{y_2} y_1 \sqrt{y_2} + \frac{1}{2} \frac{y_1}{\sqrt{y_2}} \rho \sigma \sqrt{y_2} \right) \\
 &= \mu y_1 - \frac{1}{2} \left( y_1 y_2 + \frac{1}{2} \rho \sigma y_1 \right) \\
 &= y_1 \left( \mu - \frac{1}{2} y_2 - \frac{1}{4} \rho \sigma \right) \\
 V_0^2(y) &= \kappa (\theta - y_2) - \frac{1}{2} \left( \frac{1}{2} \frac{\rho \sigma}{\sqrt{y_2}} \rho \sigma \sqrt{y_2} + \frac{\sigma \sqrt{1 - \rho^2}}{2 \sqrt{y_2}} \sigma \sqrt{1 - \rho^2} \sqrt{y_2} \right) \\
 &= \kappa (\theta - y_2) - \frac{1}{2} \left( \frac{1}{2} \rho^2 \sigma^2 + \frac{1}{2} \sigma^2 (1 - \rho^2) \right) \\
 &= \kappa (\theta - y_2) - \frac{1}{4} \sigma^2 \\
 V_0^3(y) &= y_1.
 \end{aligned}$$

So

$$\begin{aligned}
 V_0((y_1, y_2, y_3)') &= \left( y_1 \left( \mu - \frac{1}{2} y_2 - \frac{1}{4} \rho \sigma \right), \kappa (\theta - y_2) - \frac{1}{4} \sigma^2, y_1 \right)', \\
 V_1((y_1, y_2, y_3)') &= (y_1 \sqrt{y_2}, \rho \sigma \sqrt{y_2}, 0)', \\
 V_2((y_1, y_2, y_3)') &= (0, \sigma \sqrt{1 - \rho^2} \sqrt{y_2}, 0)'.
 \end{aligned}$$

As we have equations in the Stratonovich form mentioned in Equation (5.1.7), we can now introduce the implementation of the algorithm.

### Computation of $\exp(sV)$

For  $\exp(sV_1)$  and  $\exp(sV_2)$ , the ODEs obtained can be solved directly, so we avoid problems caused by numerical approximation procedures. This is not the case for  $\exp(sV_0)$  where we are obliged to use an approximation method.

#### 1. $\exp(sV_1)$

We know that  $\exp(sV_1)(y_1, y_2, y_3)$  is the solution at time  $s$  of the ODE

$$\frac{dY}{dt}(t) = V_1(Y(t)), \quad (5.1.8a)$$

$$Y(0) = (y_1, y_2, y_3), \quad (5.1.8b)$$

which is equivalent to the following system:

$$\frac{dY_1}{dt}(t) = Y_1(t) \sqrt{Y_2(t)}, \quad (5.1.9)$$

$$\frac{dY_2}{dt}(t) = \rho\sigma\sqrt{Y_2(t)}, \quad (5.1.10)$$

$$\frac{dY_3}{dt}(t) = 0. \quad (5.1.11)$$

Considering Equation (5.1.11), we have that  $Y_3(t) = c_1$ , where  $c_1$  is a constant. And by taking into account the initial condition we obtain  $c_1 = y_3$ , that is,

$$Y_3(t) = y_3.$$

Solving Equation (5.1.10), we obtain

$$Y_2(t) = \left( \max \left( 0, \frac{1}{2}\rho\sigma t + c_2 \right) \right)^2 = \left( \frac{1}{2}\rho\sigma t + c_2 \right)^2 \quad \text{for } \rho \geq 0,$$

we then use the initial condition  $Y_2(0) = y_2 > 0$  and get  $c_2 = \sqrt{y_2}$ . This gives

$$Y_2(s) = \left( \frac{1}{2}\rho\sigma s + \sqrt{y_2} \right)^2 \quad \text{where we consider } \rho \geq 0.$$

The solution to Equation (5.1.9) is given by

$$Y_1(s) = y_1 \exp \left( s\sqrt{y_2} + \frac{\rho\sigma}{4}s^2 \right).$$

We conclude that

$$\exp(sV_1)((y_1, y_2, y_3)') = \left( y_1 \exp \left( s\sqrt{y_2} + \frac{\rho\sigma}{4}s^2 \right), \left( \frac{1}{2}\rho\sigma s + \sqrt{y_2} \right)^2, y_3 \right)'. \quad (5.1.12)$$

2.  $\exp(sV_2)$

$\exp(sV_2)$  is computed in the same way as previously and we obtain

$$\exp(sV_2)((y_1, y_2, y_3)') = \left( y_1, \left( \frac{s\sigma\sqrt{1-\rho^2}}{2} + \sqrt{y_2} \right)^2, y_3 \right)'. \quad (5.1.13)$$

3.  $\exp(sV_0)$

For  $\exp(sV_0)$  there is no closed solution, we use an approximation in order to compute it.

We have  $V_0((y_1, y_2, y_3)') = (y_1(\mu - \frac{1}{2}y_2 - \frac{1}{4}\rho\sigma), \kappa(\theta - y_2) - \frac{1}{4}\sigma^2, y_1)'$ .  
Let us consider the following system of ODEs:

$$\frac{dY_1}{dt}(t) = Y_1(t) \left( \mu - \frac{Y_2(t)}{2} - \frac{\rho\sigma}{2} \right), \quad (5.1.14)$$

$$\frac{dY_2}{dt}(t) = \kappa(\theta - Y_2(t)) - \frac{\sigma^2}{4}, \quad (5.1.15)$$

$$\frac{dY_3}{dt}(t) = Y_1(t). \quad (5.1.16)$$

We first solve the first-order ordinary differential (5.1.15) and obtain

$$Y_2(t) = (y_2 - J) \exp(-\kappa t) + J \quad \text{where} \quad J = \theta - \frac{\sigma^2}{4\kappa}.$$

We now consider Equation (5.1.14),

$$\begin{aligned} \frac{dY_1}{dt} &= Y_1 \left( \mu - \frac{Y_2}{2} - \frac{\rho\sigma}{4} \right) \\ &= Y_1 \left( \mu - \frac{J}{2} - \frac{1}{2}(y_2 - J) \exp(-\kappa t) - \frac{\rho\sigma}{4} \right); \end{aligned}$$

when solving, we obtain

$$Y_1(t) = c \cdot \exp \left[ \left( \mu - \frac{J}{2} - \frac{\rho\sigma}{4} \right) t + \frac{1}{2\kappa} (y_2 - J) \exp(-\kappa t) \right],$$

where  $c$  is a constant. Taking into account the initial condition  $y(0) = y_1$ , we obtain

$$c = y_1 \exp \left( -\frac{1}{2\kappa} (y_2 - J) \right),$$

that is,

$$Y_1(t) = y_1 \cdot \exp \left[ \left( \mu - \frac{J}{2} - \frac{\rho\sigma}{4} \right) t + \frac{1}{2\kappa} (y_2 - J) (\exp(-\kappa t) - 1) \right]. \quad (5.1.17)$$

In order to solve Equation (5.1.16), we expand  $\exp(-\kappa t)$  in (5.1.17) and we obtain

$$\begin{aligned} Y_1(t) &= y_1 \cdot \exp \left[ \left( \mu - \frac{J}{2} - \frac{\rho\sigma}{4} \right) t + \frac{1}{2\kappa} (y_2 - J) (1 - \kappa t + \mathcal{O}(t^2) - 1) \right], \\ &= y_1 \cdot \exp \left[ \left( \mu - \frac{y_2}{2} - \frac{\rho\sigma}{4} \right) t \right] + \mathcal{O}(t^2), \end{aligned}$$

which implies that Equation (5.1.16) is

$$\frac{dY_3}{dt} = y_1 e^{At} + \mathcal{O}(t^2)$$

where  $A = \mu - \frac{y_2}{2} - \frac{\rho\sigma}{4}$ . Therefore

$$Y_3(t) = y_3 + \frac{y_1}{A} (e^{At} - 1) + \mathcal{O}(t^3).$$

We conclude that

$$\exp(sV_0) ((y_1, y_2, y_3)') = (g_1(s), g_2(s), g_3(s))', \quad (5.1.18)$$

where

$$g_1(s) = y_1 \cdot \exp \left[ \left( \mu - \frac{J}{2} - \frac{\rho\sigma}{4} \right) s + \frac{1}{2\kappa} (y_2 - J) (e^{-\kappa s} - 1) \right] \quad (5.1.19)$$

$$g_2(s) = (y_2 - J) e^{-\kappa s} + J, \quad (5.1.20)$$

$$g_3(s) = y_3 + \frac{y_1}{A} (e^{As} - 1) + \mathcal{O}(s^3), \quad (5.1.21)$$

with

$$J = \theta - \frac{\sigma^2}{4\kappa}, \quad \text{and} \quad A = \mu - \frac{y_2}{2} - \frac{\rho\sigma}{4}. \quad (5.1.22)$$

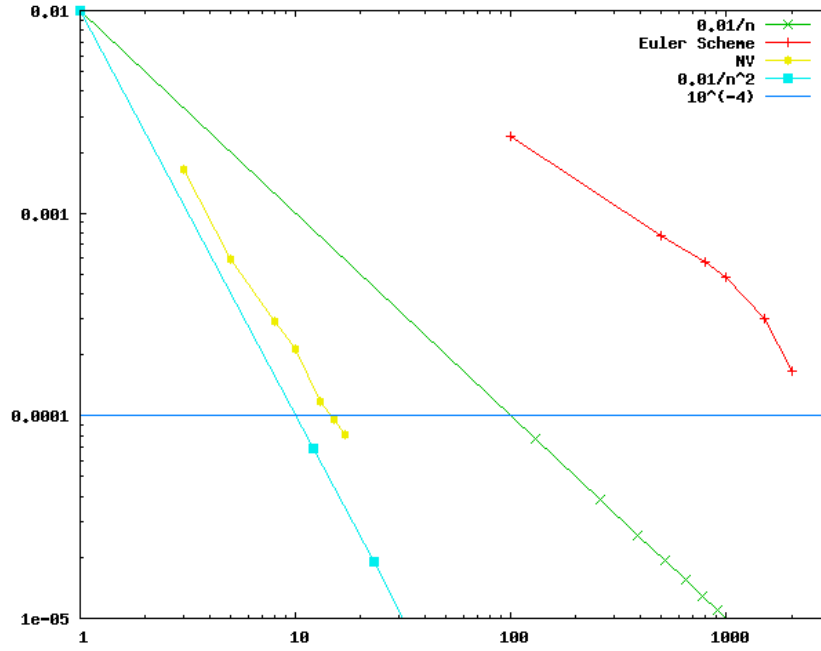
The new random variable  $\bar{X}_{(n)}$  is then constructed as follows:

$$\begin{aligned} \bar{X}_{(0)} &= x, \\ \bar{X}_{(k+1)} &= \begin{cases} \exp(s_0 V_0) \exp(s_1^k V_1) \exp(s_2^k V_2) \exp(s_0 V_0) \bar{X}_{(k)} & \text{if } \Lambda_k = +1, \\ \exp(s_0 V_0) \exp(s_2^k V_2) \exp(s_1^k V_1) \exp(s_0 V_0) \bar{X}_{(k)} & \text{if } \Lambda_k = -1. \end{cases} \end{aligned}$$

where  $\exp(s_i V_i) x$  is computed as previously,  $s_0 = \frac{T}{2n}$ ,  $s_1^k = \sqrt{\frac{T}{n}} \zeta_k^1$  and  $s_2^k = \sqrt{\frac{T}{n}} \zeta_k^2$  with  $(\zeta_j^i)_{i,j}$  being independent standard normal random variables. We can now use Monte Carlo or Quasi-Monte Carlo method in order to compute  $E[f(\bar{X}_{(n)}(T, x))]$ . We compare the results of the above implementation with the Classical Euler-Maruyama algorithm. We consider as control value,

$$E[\max(X_3(T, x)/T - K, 0)] = 0.0604194813$$

obtained by the new algorithm with extrapolation, number of discretization points  $n = 150$  and number of sample points for Monte Carlo  $M = 4.0 \times 10^4$ . We set  $T = 1$ ,  $K = 1.05$ ,  $\mu = 0.05$ ,  $\kappa = 2.0$ ,  $\sigma = 0.1$ ,  $\theta = 0.09$ ,  $\rho = 0$  and  $x = (1.0, 0.09, 0)$ .



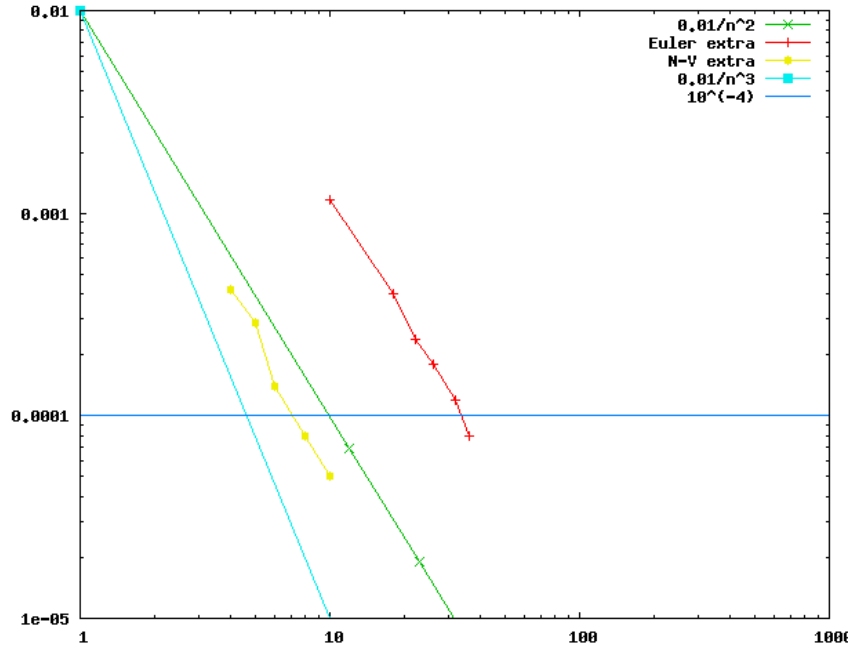
**Figure 5.1:** Comparison of the Ninomiya-Victoir and Euler schemes.

Figure 5.1 illustrates the relation between the error due to the two approximations schemes and the numbers of points needed for the discretization. We observe the following:

To achieve an order 4 accuracy, we only need  $n = 13$  when using the new algorithm (Ninomiya-Victoir), whereas in the Euler-Maruyama scheme  $n$  has to be greater than 2000.

We also obtain that the new algorithm (computation time =  $1.8 \times 10^2$  seconds) is about 60 times faster than the traditional Euler method (computation time =  $1.1 \times 10^4$  seconds).





**Figure 5.2:** Comparison of the Ninomiya-Victoir and Euler schemes with extrapolation.

In order to reduce the number of points needed for the simulations, we consider the two algorithms with Romberg extrapolation.

The result is presented in Figure 5.2, from which we observe that the number of discretization points needed to achieve an order 4 accuracy is 7 in the case of the Ninomiya-Victoir algorithm whereas we need about 32 points for the Euler algorithm. We also obtained that the Ninomiya-Victoir algorithm without extrapolation is still faster and needs less points than the Euler-Maruyama method with extrapolation. A summary of these observations is given in Table 5.1 .

**Table 5.1:** Results of simulations (to achieve order 4 accuracy.)

Algorithm	Number of points	Computation time (sec)
Euler	2000	$1.1 \times 10^4$
Ninomiya-Victoir	13	$1.8 \times 10^2$
Euler + Extrapolation	32	$2.56 \times 10^3$
Nino-Victoir + Extra	7	81.18

### 5.1.2.2 Optimal Portfolio Strategies

As for many financial problems, it is difficult to obtain an analytical solution and one is then obliged to use numerical methods. In this section, we consider a Stock-Bond-Cash allocation problem introduced by Fukaya (2006) where he constructed an optimal solution to this problem as an expected value of some Markovian-types diffusion processes. Let's denote by  $\hat{\varphi}(t)$  the solution at time  $t$  to this problem. Based on ideas presented in Fukaya (2005), we derive a general expression for  $\hat{\varphi}(0)$  and then implement the Ninomiya-Victoir algorithm for its computation.

#### 1. Settings and Derivation of $\hat{\varphi}(0)$

Without going into detail around the mathematical background of the derivation of  $\hat{\varphi}(0)$ , we can outline the general settings and results as follows:

Consider a market modelled by an Ornstein-Uhlenbeck process  $\{S(t)\}_{0 \leq t \leq T}$  satisfying

$$dS(t, x_0) = -aS(t, x_0)dt + b dB^1(t),$$

that is  $S(t, x_0) = x_0 e^{-at} + b e^{-at} \int_0^t e^{as} dB^1(s)$ , where  $a > 0, b \neq 0$ ,  $B^1(t)$  is a one-dimensional Brownian motion and  $x_0$  is the initial value of the process. The utility function is considered to be of power type  $(\gamma, \beta, 0, 0)$  and the risk-free rate is a function of time and is given at time  $t$  by

$$r(t, x_0) = r(S(t, x_0)) = c \left( \log \left( 1 + e^{S(t, x_0)} \right) \right)^\alpha \quad (5.1.23)$$

for some  $\alpha \in (0, 1)$  and  $c > 0$ .

We introduce the following processes:

- The risk-free asset also called money account  $S_0(t, x_0)$  is defined by

$$S_0(t, x_0) = \exp \left\{ \int_0^t r(s, x_0) ds \right\}. \quad (5.1.24)$$

- The stock price process  $S_1(t, x_0)$  satisfying

$$S_1(t, x_0) = s_1 \exp \left\{ \left( \mu - \frac{1}{2}(\rho^2 + \sigma^2) \right) t + \rho B^1(t) + \sigma B^2(t) \right\}, \quad (5.1.25)$$

where  $\sigma > 0, \rho \neq 0$  and  $B^2(t)$  is a one-dimensional Brownian motion independent of  $B^1(t)$ .

- A zero bond process  $S_2(t, x_0)$  modelled by

$$S_2(t, x_0) = \mathbb{E}^{\mathbb{Q}} \left[ \exp \left\{ - \int_t^T r(s, x_0) ds \right\} \middle| \mathcal{F}_t \right], \quad (5.1.26)$$

where  $\mathbb{Q}$  is the risk neutral probability measure considered to be defined by the market price of risk processes  $(\alpha_1(t, x_0), \alpha_2(t, x_0))$  given here by

$$\alpha_1(t, x_0) = \alpha_1 = \text{constant}; \quad \alpha_2(t, x_0) = c_1 - c_2 r(t, x_0),$$

where  $c_1 = (\mu - \rho\alpha_1)/\sigma$  and  $c_2 = 1/\sigma$ .

- The state price density process is

$$\begin{aligned} \Pi(t, x) = & \exp \left\{ - \int_0^t r(s, x) ds - \int_0^t \alpha_1 dB^1(s) - \int_0^t (c_1 - c_2 r(s, x)) dB^2(s) \right. \\ & \left. - \frac{1}{2} \int_0^t (\alpha_1^2 + (c_1 - c_2 r(s, x))^2) ds \right\}. \end{aligned} \quad (5.1.27)$$

The previous equation is equivalent to

$$d\Pi(t, x) = \Pi(t, x) (-r(t, x) dt - \alpha_1 dB^1(t) - (c_1 - c_2 r(t, x)) dB^2(t)).$$

We also introduce the process

$$\pi(t, x) = - \int_0^t \{c_1 c_2 - 1 - c_2^2 r(s, x)\} r'(s, x) e^{-as} ds + c_2 \int_0^t r'(s, x) e^{-as} dB^2(s). \quad (5.1.28)$$

The volatility matrix of  $S_1(t, x)$  and  $S_2(t, x)$  at time 0 is given by

$$\varpi(0, x) = \begin{pmatrix} \rho & \sigma \\ \sigma_2(x) & 0 \end{pmatrix},$$

where

$$\sigma_2(x) = \frac{b}{S_2(0, x)} \cdot \frac{\partial S_2}{\partial x}(0, x), \quad (5.1.29)$$

with

$$S_2(0, x) = \mathbb{E}[\Pi(T, x)] \quad \text{and} \quad \frac{\partial S_2}{\partial x}(0, x) = \mathbb{E}[\pi(T, x) \Pi(T, x)].$$

Assuming that the investor has a utility function of power type  $(\gamma, \beta, 0, 0)$  where  $\gamma \in (0, 1)$  and  $\beta > 0$ , we introduce the following:

$$\Theta(t, x) = \exp \left\{ \beta_1 \int_0^t r(s, x) ds \right\} \quad (5.1.30)$$

$$\begin{aligned} \Delta(t, x) = & \exp \left\{ \beta_2 \int_0^t r(s, x) ds \right\} \times \exp \left\{ \beta_3 \left( \left( \mu - \frac{1}{2} (\rho^2 + \sigma^2) \right) t \right. \right. \\ & \left. \left. + \rho B^1(t) + \sigma B^2(t) \right) \right\} \end{aligned} \quad (5.1.31)$$

$$\eta(t, x) = \beta_1 \int_0^t r'(s, x) e^{-as} ds \quad (5.1.32)$$

$$\delta(t, x) = \beta_2 \int_0^t r'(s, x) e^{-as} ds. \quad (5.1.33)$$

Having defined all the above processes, we have the following result for the optimal portfolio strategy at time  $t = 0$  taken from Fukaya (2005):

$$\hat{\varphi}(0) = \begin{pmatrix} \rho_s \\ \rho_b \end{pmatrix} = \frac{1}{\gamma} \begin{pmatrix} \frac{1}{\sigma} (c_1 - c_2 r(0, x_0)) - \frac{\beta_3 A_2(0)}{\beta^{1/\gamma} A_1(0) + A_2(0)} \\ \frac{\alpha_1}{\sigma_2(x_0)} - \frac{\rho(c_1 - c_2 r(0, x_0))}{\sigma \sigma_2(x_0)} - \frac{b}{\sigma_2(x_0)} \frac{D(0)}{\beta^{1/\gamma} A_1(0) + A_2(0)} \end{pmatrix}, \quad (5.1.34)$$

where  $\rho_s$  is the optimal portfolio strategy on the stock,  $\rho_b$  is the optimal portfolio strategy on the bond and

$$A_1(0) = \mathbb{E} \left[ \int_0^{T_0} \Pi(s, x_0)^{1-1/\gamma} \Theta(s, x_0)^{-1/\gamma} ds \right] \quad (5.1.35)$$

$$A_2(0) = \mathbb{E} \left[ \Pi(T_0, x_0)^{1-1/\gamma} \Delta(T_0, x_0)^{-1/\gamma} \right] \quad (5.1.36)$$

$$\begin{aligned} D(0) &= (1 - \gamma) \mathbb{E} \left[ \beta^{1/\gamma} \int_0^{T_0} \pi(s, x_0) \Pi(s, x_0)^{1-1/\gamma} \Theta(s, x_0)^{-1/\gamma} ds \right. \\ &\quad \left. + \pi(T_0, x_0) \Pi(T_0, x_0)^{1-1/\gamma} \Delta(T_0, x_0)^{-1/\gamma} \right] \\ &\quad + \mathbb{E} \left[ \beta^{1/\gamma} \int_0^{T_0} \eta(s, x_0) \Pi(s, x_0)^{1-1/\gamma} \Theta(s, x_0)^{-1/\gamma} ds \right. \\ &\quad \left. + \delta(T_0, x_0) \Pi(T_0, x_0)^{1-1/\gamma} \Delta(T_0, x_0)^{-1/\gamma} \right] \end{aligned} \quad (5.1.37)$$

and  $\sigma_2(x)$  is defined as in (5.1.29).

## 2. Implementation of the Ninomiya-Victoir Algorithm

We now apply the Ninomiya-Victoir scheme for the numerical computation of  $\hat{\varphi}(0)$ . Numerical results for the stock are presented. We first set up a system SDEs derived from the expression of  $\hat{\varphi}(0)$ .

We consider

$$\begin{aligned} X(t, x_0) &= \left( S(t, x_0), \Pi(t, x_0), \pi(t, x_0), \Delta(t, x_0), \delta(t, x_0), \right. \\ &\quad \left. \Theta(t, x_0), \eta(t, x_0), G(t, x_0), H(t, x_0), I(t, x_0), e^{-at} \right)' \end{aligned}$$

where

$$G(t, x_0) = \int_0^t \Pi(s, x_0)^{1-1/\gamma} \Theta(s, x_0)^{-1/\gamma} ds$$

is the term appearing in (5.1.35),

$$H(t, x_0) = \int_0^t \pi(s, x_0) \Pi(s, x_0)^{1-1/\gamma} \Theta(s, x_0)^{-1/\gamma} ds \quad \text{and}$$

$$I(t, x_0) = \int_0^t \eta(s, x_0) \Pi(s, x_0)^{1-1/\gamma} \Theta(s, x_0)^{-1/\gamma} ds \quad \text{are the terms}$$

appearing in (5.1.37). The other processes are defined as previously.

We then obtain a system of stochastic differential equations which in Itô form is written as

$$dX(t, x) = \sum_{j=0}^2 \bar{V}_j(X(t, x)) dB^j(t)$$

where  $dB^0(t) = dt$  and the vectors field  $\bar{V}_0, \bar{V}_1, \bar{V}_2$  are defined from  $\mathbb{R}^{11}$  into  $\mathbb{R}^{11}$  as follows:

$$\begin{aligned} \bar{V}_0^1(X) &= -aX_1, & \bar{V}_0^2(X) &= -r(X_1)X_2, \\ \bar{V}_0^3(X) &= -\{c_1c_2 - 1 - c_2^2r(X_1)\}r'(X_1)X_{11}, \\ \bar{V}_0^4(X) &= \left(\beta_3\mu + \beta_2r(X_1) + \frac{1}{2}\beta_3(\rho^2 + \sigma^2)(\beta_3 - 1)\right)X_4, \\ \bar{V}_0^5(X) &= \beta_2X_{11}r'(X_1), & \bar{V}_0^6(X) &= \beta_1X_6r(X_1), & \bar{V}_0^7(X) &= \beta_1r'(X_1)X_{11}, \\ \bar{V}_0^8(X) &= X_2^{1-1/\gamma}X_6^{-1/\gamma}, & \bar{V}_0^9(X) &= X_3X_2^{1-1/\gamma}X_6^{-1/\gamma}, \\ \bar{V}_0^{10}(X) &= X_7X_2^{1-1/\gamma}X_6^{-1/\gamma}, & \bar{V}_0^{11}(X) &= -aX_{11} \end{aligned}$$

$$\begin{aligned} \bar{V}_1^1(X) &= b, & \bar{V}_1^2(X) &= -\alpha_1X_2, & \bar{V}_1^3(X) &= 0, \\ \bar{V}_1^4(X) &= \rho\beta_3X_4, & \bar{V}_1^5(X) &= \dots = \bar{V}_1^{11}(X) = 0 \end{aligned}$$

$$\begin{aligned} \bar{V}_2^1(X) &= 0, & \bar{V}_2^2(X) &= -(c_1 - c_2r(X_1))X_2, & \bar{V}_2^3(X) &= c_2r'(X_1)X_{11}, \\ \bar{V}_2^4(X) &= \beta_3\sigma X_4, & \bar{V}_2^5(X) &= \dots = \bar{V}_2^{11}(X) = 0. \end{aligned}$$

When transforming the system into Stratonovich form, we have

$$dX(t, x) = \sum_{j=0}^2 V_j(X(t, x)) \circ dB^j(t) \quad (5.1.38)$$

where  $V_1 = \bar{V}_1, V_2 = \bar{V}_2$  and  $V_0$  is computed as in the previous section and is given by

$$\begin{aligned} V_0^1(X) &= -aX_1, & V_0^2(X) &= \left(-r(X_1) - \frac{1}{2}\alpha_1^2 - \frac{1}{2}(c_1 - c_2r(X_1))^2\right)X_2, \\ V_0^3(X) &= -\{c_1c_2 - 1 - c_2^2r(X_1)\}r'(X_1)X_{11}, \\ V_0^4(X) &= \left(\beta_3\mu + \beta_2r(X_1) - \frac{1}{2}\beta_3\rho^2 - \frac{1}{2}\beta_3\sigma^2\right)X_4, \\ V_0^5(X) &= \beta_2r'(X_1)X_{11}, & V_0^6(X) &= \beta_1r(X_1)X_6, & V_0^7(X) &= \beta_1r'(X_1)X_{11}, \\ V_0^8(X) &= X_2^{1-1/\gamma}X_6^{-1/\gamma}, & V_0^9(X) &= X_3X_2^{1-1/\gamma}X_6^{-1/\gamma}, \\ V_0^{10}(X) &= X_7X_2^{1-1/\gamma}X_6^{-1/\gamma}, & V_0^{11}(X) &= -aX_{11}. \end{aligned}$$

The next step is to compute the solution of our system of stochastic differential equations (5.1.38) using the Ninomiya-Victoir algorithm. In the following, we compute  $\exp(sV)$ .

- $\exp(sV_1)(x)$

We know from the previous section that  $\exp(sV_1)(x)$  is the solution at time  $s$  of the ODE

$$\begin{aligned}\frac{dY}{dt}(t) &= V_1(Y(t)) \\ Y(0) &= x.\end{aligned}$$

It is equivalent to the following system of equations:

$$\frac{dY_1}{dt}(t) = b, \quad (5.1.39)$$

$$\frac{dY_2}{dt}(t) = -\alpha_1 Y_2(t), \quad (5.1.40)$$

$$\frac{dY_3}{dt}(t) = 0, \quad (5.1.41)$$

$$\frac{dY_4}{dt}(t) = \rho Y_4(t) \beta_3, \quad (5.1.42)$$

$$\frac{dY_5}{dt}(t) = \dots = \frac{dY_{11}}{dt}(t) = 0. \quad (5.1.43)$$

The solutions to these equations are given, for  $i \in \{3, 5, 6, 7, 8, 9, 10, 11\}$ , by  $Y_i(t) = x_i$ .

Considering the Equation (5.1.39), we have

$$Y_1(t) = bt + c.$$

Taking the initial condition into account, we obtain  $c = x_1$ , so

$$Y_1(t) = bt + x_1.$$

We also have the following result for Equation (5.1.40) and (5.1.42).

$$Y_2(t) = x_2 e^{-\alpha_1 t} \quad \text{and} \quad Y_4(t) = x_4 e^{\rho \beta_3 t}.$$

The result can be summarized as

$$\exp(sV_1)(x) = \begin{pmatrix} 1 & & & & & & & & & & 0 \\ & e^{-\alpha_1 s} & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & e^{\rho \beta_3 s} & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ 0 & & & & & & & & & & 1 \end{pmatrix} x + \begin{pmatrix} bs \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (5.1.44)$$

- $\exp(sV_2)(x)$

Performing a similar calculation as for  $\exp(sV_1)(x)$ , we obtain

$$\begin{aligned} \exp(sV_2)(x) = & \begin{pmatrix} 1 & & & & 0 \\ & e^{-(c_1 - c_2 r(x_1))s} & & & \\ & & 1 & & \\ & & & e^{\sigma\beta_3 s} & \\ & & & & 1 \\ 0 & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} x + \\ & + \begin{pmatrix} 0 \\ 0 \\ c_2 r'(x_1) x_{11} s \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned} \quad (5.1.45)$$

One can easily check that

$$\exp(sV_1) \circ \exp(sV_2) = \exp(sV_2) \exp(sV_1).$$

Therefore, Theorem 5.1.1 can be stated as follows:

**Proposition 5.1.3.** *Let  $(\zeta_i)_{i \in \{1, \dots, n\}}$  be  $n$  independent standard 2-dimensional normal random variables. The family of random variable  $\{\bar{X}_{(k)}\}_{k=0, \dots, n}$  is defined as follows:*

$$\begin{aligned} \bar{X}_{(0)} &= x, \\ \bar{X}_{(k+1)} &= \exp\left(\frac{T}{2n} V_0\right) \exp\left(\sqrt{\frac{T}{n}} \zeta_k^1 V_1\right) \exp\left(\sqrt{\frac{T}{n}} \zeta_k^2 V_2\right) \exp\left(\frac{T}{2n} V_0\right) \bar{X}_{(k)}. \end{aligned}$$

Then, we have an order 2 approximation.

- $\exp(sV_0)(x)$

Considering the ODE

$$\begin{aligned} \frac{dY}{dt}(t) &= V_0(Y(t)) \\ Y(0) &= x, \end{aligned} \quad (5.1.46)$$

we use the 2-stage 2-order Runge-Kutta method taken from Butcher (2008) in order to compute  $\exp(sV_0)(x)$  which is the solution at time

$s$  of the ODEs (5.1.46). For small  $s$ , the 2-stage 2-order Runge-Kutta method is written as

$$\exp(sV_0)(x) = Y(s) \simeq x + \frac{1}{2}(k_1 + k_2), \quad \text{where } k_1 = sV_0(x) \quad \text{and}$$

$k_2 = sV_0(x + k_1)$ , that is

$$\exp(sV_0)(x) \simeq x + \frac{1}{2}(sV_0(x) + sV_0(x + sV_0(x))).$$

We have now succeeded to compute  $\exp(sV_i)_{i=0,1,2}$ . Considering the expression of  $X(t, x_0)$ , we have the following expressions for  $S^2(0, x_0)$ ;  $\frac{\partial S^2}{\partial x}(0, x_0)$ ;  $A_1(0)$ ;  $A_2(0)$  and  $D(0)$ :

$$\begin{aligned} S^2(0, x_0) &= \mathbb{E}[X_2(T, x)], \\ \frac{\partial S^2}{\partial x}(0, x_0) &= \mathbb{E}[X_3(T, x) X_2(T, x)], \\ A_1(0) &= \mathbb{E}[X_8(T_0, x)], \\ A_2(0) &= \mathbb{E}\left[X_2(T_0, x)^{1-1/\gamma} X_4(T_0, x)^{-1/\gamma}\right], \\ D(0) &= (1 - \gamma) \left\{ \beta^{1/\gamma} \mathbb{E}[X_9(T_0, x)] \right. \\ &\quad \left. + \mathbb{E}\left[X_3(T_0, x) X_2(T_0, x)^{1-1/\gamma} X_4(T_0, x)^{-1/\gamma}\right] \right\} \\ &\quad + \beta^{1/\gamma} \mathbb{E}[X_{10}(T_0, x)] + \mathbb{E}\left[X_5(T_0, x) X_2(T_0, x)^{1-1/\gamma} X_4(T_0, x)^{-1/\gamma}\right]. \end{aligned}$$

We now introduce the function  $\phi$  defined from  $\mathbb{R}^{11} \rightarrow \mathbb{R}^5$  by  $\phi(x) = (\phi_1(x), \dots, \phi_5(x))$  where

$$\begin{aligned} \phi_1(x) &= x_2; \quad \phi_2(x) = x_3 x_2; \quad \phi_3(x) = x_8; \quad \phi_4(x) = x_2^{1-1/\gamma} x_4^{-1/\gamma}; \\ \phi_5(x) &= (1 - \gamma) \left\{ \beta^{1/\gamma} x_9 + x_3 x_2^{1-1/\gamma} x_4^{-1/\gamma} \right\} + \left\{ \beta^{1/\gamma} x_{10} + x_5 x_2^{1-1/\gamma} x_4^{-1/\gamma} \right\}. \end{aligned}$$

We can now perform the computation of  $\phi(\bar{X}_{(n)})$  where  $\bar{X}_{(n)}$  is constructed as in Proposition 5.1.3 and then use Monte Carlo methods to obtain the final result.

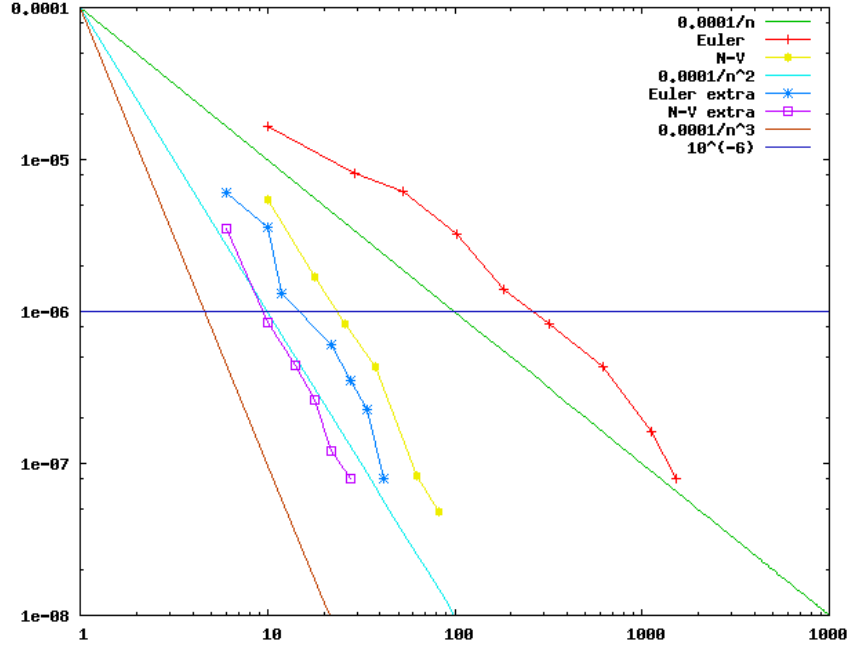
## Simulation Results

For the numerical simulation, we consider the fixed parameters given in Table 5.2. In this experiment, we consider as control value  $\rho_s = 1.20798645$  for the stock which is obtained by the Ninomiya-Victoir algorithm with extrapolation,  $n = 1500$  discretization points and  $M = 2 \times 10^4$ .



**Table 5.2:** Fixed parameters

$x_0$	$a$	$b$	$\alpha$	$c$	$\mu$	$\rho$	$\sigma$
1.0	1.0	2.0	0.9	0.01	0.08	-0.14	0.20
$T$	$T_0$	$\alpha_1$	$\gamma$	$\beta$	$\beta_1$	$\beta_2$	$\beta_3$
2.0	1.0	-0.165	0.9	2.0	0.1	0.05	0.05



**Figure 5.3:** Error coming from the discretization: Stock.

As in the case of an option pricing problem, Figure 5.3 shows the relation between the number of partitions and the errors of the methods. We observe that the Ninomiya-Victoir method requires  $n = 10$  and  $n = 24$  when considered with and without extrapolation, respectively, while the Euler-Maruyama scheme with extrapolation needs  $n = 14$  and the simple Euler-Maruyama method requires about  $n = 220$  discretization points. A summary of the computation time to achieve order 6 accuracy is given in Table 5.3. From the table, we observe that the Ninomiya-Victoir method as in the previous example is much faster than the Euler-Maruyama method. However, notice that in this example, some  $V_j$  are not elements of  $C_b^\infty(\mathbb{R}^N)$ . Therefore this problem does not satisfy conditions of the Ninomiya-Victoir approximation scheme but this implementation shows that there is a great need of extension of the Ninomiya-Victoir approximation scheme under a much weaker hypothesis on the vector fields  $V_j$ .

**Table 5.3:** Results of simulations (to achieve order 6 accuracy.)

Algorithm	Number of points	Computation time (sec)
Euler	220	$1.1 \times 10^4$
Ninomiya-Victoir	24	$8.12 \times 10^2$
Euler + Extrapolation	14	$7.56 \times 10^3$
Nino-Victoir + Extra	10	$5.45 \times 10^2$

## 5.2 Ninomiya–Ninomiya Scheme

In this last Section, we present another new approximation scheme which is the algorithm constructed by Ninomiya and Ninomiya, 2009. An intuitive explanation of the scheme is as follows: We construct an ODE-valued random variable whose average approximates the solution of a given SDE. From this new random variable, an ODE is then drawn and the conventional Runge-Kutta method is applied in order to approximate the ODE. We illustrate this approximation scheme by applying the algorithm to the problem of pricing the Asian option presented in Subsection 5.1.2.1.

### 5.2.1 Notations

Let  $\mathcal{A}$  be the set of all multi-indices. We define the  $\mathbb{R}$ -algebra of formal series and the free  $\mathbb{R}$ -algebra with basis  $\mathcal{A}$  by  $\mathbb{R}\langle\langle\mathcal{A}\rangle\rangle := \{X = \sum_{\alpha \in \mathcal{A}} x_\alpha \alpha \mid x_\alpha \in \mathbb{R}\}$  and  $\mathbb{R}\langle\mathcal{A}\rangle := \{X = \sum_{\alpha \in \mathcal{A}} x_\alpha \alpha \in \mathbb{R}\langle\langle\mathcal{A}\rangle\rangle \mid \exists k \in \mathbb{N} \text{ such that } x_\alpha = 0 \text{ if } |\alpha| \geq k\}$ , respectively. Then,  $\mathbb{R}\langle\mathcal{A}\rangle$  is a sub- $\mathbb{R}$ -algebra of  $\mathbb{R}\langle\langle\mathcal{A}\rangle\rangle$ . Let  $\mathbb{R}\langle\mathcal{A}\rangle_m = \{X \in \mathbb{R}\langle\mathcal{A}\rangle \mid x_\alpha = 0, \text{ if } \|\alpha\| \neq m\}$ . For  $X, Y \in \mathbb{R}\langle\langle\mathcal{A}\rangle\rangle$ , we define the algebra structure as

$$Z = X.Y = \left( \sum_{\alpha \in \mathcal{A}} x_\alpha \alpha \right) \left( \sum_{\beta \in \mathcal{A}} y_\beta \beta \right) = \sum_{\gamma \in \mathcal{A}} z_\gamma \gamma,$$

where

$$z_\gamma = \sum_{\substack{\gamma = \alpha\beta \\ \alpha, \beta \in \mathcal{A}}} x_\alpha y_\beta,$$

and the Lie bracket by  $[X, Y] = XY - YX$ . For all  $X \in \mathbb{R}\langle\langle\mathcal{A}\rangle\rangle$ , set

$$X|_k := \sum_{|\alpha|=k} x_\alpha \alpha$$

and

$$\mathcal{J}_{\mathcal{A}} := \left\{ K \subset \mathbb{R}\langle\mathcal{A}\rangle \mid \mathcal{A} \subset K \text{ and } [X, Y] \in K, \text{ for all } X, Y \in K \right\}.$$

We define  $\mathbf{L}(\mathcal{A}) := \bigcap_{K \in \mathcal{J}_{\mathcal{A}}} K$  as the set of Lie polynomials in  $\mathbb{R}\langle\mathcal{A}\rangle$  and

$$\mathbf{L}(\langle\mathcal{A}\rangle) := \left\{ X \in \mathbb{R}\langle\langle\mathcal{A}\rangle\rangle \quad \text{such that} \quad X|_k \in \mathbf{L}(\mathcal{A}), \forall k \in \mathbb{N} \right\}$$

as the set of Lie series.

We introduce the following definition:

**Definition 5.2.1.** • For  $m \in \mathbb{N}$ ,  $j_m$  is the map defined by

$$j_m \left( \sum_{\alpha \in \mathcal{A}} x_{\alpha} \alpha \right) = \sum_{\|\alpha\| \leq m} x_{\alpha} \alpha.$$

- Let's denote by  $\mathbf{A}_V$  the  $\mathbb{R}$ -algebra consisting of smooth differential operators over  $\mathbb{R}^N$ . We define the homomorphism  $\phi$  from  $\mathbb{R}\langle\mathcal{A}\rangle$  to  $\mathbf{A}_V$  as

$$\begin{aligned} \phi(X) &= Id, & \text{for } X = \emptyset \\ \phi(X) &= V_{i_1} \cdots V_{i_k} & \text{for } X = \alpha = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}) \end{aligned}$$

and

$$\phi(X) = \sum_{\alpha \in \mathcal{A}} x_{\alpha} \phi(\alpha) \quad \text{for } X = \sum_{\alpha \in \mathcal{A}} x_{\alpha} \alpha.$$

- For  $s \in \mathbb{R}_+^*$ , we define the rescaling operator  $\psi_s : \mathbb{R}\langle\langle\mathcal{A}\rangle\rangle \rightarrow \mathbb{R}\langle\langle\mathcal{A}\rangle\rangle$  as

$$\psi_s \left( \sum_{m=0}^{\infty} X_m \right) = \sum_{m=0}^{\infty} s^{\frac{m}{2}} X_m,$$

where  $X_m \in \mathbb{R}\langle\mathcal{A}\rangle_m$ .

We illustrate the use of the above defined operators in the following example:

**Example 5.2.2.** Consider the computation of  $\phi(\psi_s(\alpha_0 + \frac{1}{2}\alpha_1 + \frac{1}{3}\alpha_2))$ . According to the definition of  $\psi_s$ ,

$$\psi_s \left( \alpha_0 + \frac{1}{2}\alpha_1 + \frac{1}{3}\alpha_2 \right) = s\alpha_0 + \frac{1}{2}\sqrt{s}\alpha_1 + \frac{1}{3}\sqrt{s}\alpha_2,$$

so

$$\phi \left( \psi_s \left( \alpha_0 + \frac{1}{2}\alpha_1 + \frac{1}{3}\alpha_2 \right) \right) = sV_0 + \frac{1}{2}\sqrt{s}V_1 + \frac{1}{3}\sqrt{s}V_2.$$

For a smooth vector field  $V$  we define the following norms:

$$\begin{aligned}\|V\| &= \sup \{|V(x)|; x \in \mathbb{R}^N\}, \\ \|V^{(n)}\| &= \sup \left\{ |V_{(x)}^{(n)}(a_1, a_2, \dots, a_n)|; x \in \mathbb{R}^N \text{ and } |a_i| = 1, \text{ for } i = 1, \dots, n \right\}, \\ \|V\|_{c^n} &= \sum_{i=0}^n \|V^{(i)}\|,\end{aligned}$$

where  $V_{(x)}^{(n)}(a_1, a_2, \dots, a_n) = \sum_{i=1}^N \sum_{j_1=1}^N \cdots \sum_{j_n=1}^N \frac{\partial^n V_i}{\partial x_{j_1} \cdots \partial x_{j_n}}(x) a_1^{j_1} \cdots a_n^{j_n} e_i$ , with each  $e_i$  being an  $N$ -dimensional unit vector, and  $a_k^j$  is the  $j$ th component of  $a_k \in \mathbb{R}^N$ .

Based on these norms, we introduce the following definition.

**Definition 5.2.3.** *An application  $g$  from  $C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$  to the set of all functions from  $\mathbb{R}^N$  to  $\mathbb{R}^N$  is called an integration scheme of order  $m$  if the following condition is satisfied: There exists a positive constant  $c_m$  such that*

$$\sup_{x \in \mathbb{R}^N} |g(V)(x) - \exp(V)(x)| \leq c_m \|V\|_{c^{m+1}}^{m+1},$$

for all  $V \in C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$ . We denote by  $\mathcal{IS}(m)$  the set of all integration schemes of order  $m$ .

$g(V)(x)$  defined above can be viewed as the approximation of  $\exp(V)(x)$ . For  $V \in C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$ ,  $\exp(tV)(x)$  is the solution at time  $t$  of the ODE

$$\begin{aligned}\frac{dY}{dt}(t, x) &= V(Y(t, x)), \\ Y(0, x) &= x.\end{aligned}\tag{5.2.1}$$

The construction of an integration scheme of order  $m$  is then equivalent to finding an  $m$ th order solution at time 1 to the ODE (5.2.1).

### 5.2.2 Result of the Ninomiya-Ninomiya Scheme

Before we state the results of the Ninomiya-Ninomiya scheme, we introduce the definition of the Hausdorff product.

**Definition 5.2.4.** *Let  $X_1, X_2 \in \mathbf{L}((\mathcal{A}))$ , we define the Hausdorff product of  $X_1, X_2$  as follows:*

$$X_1 \vdash X_2 := \log(\exp(X_1) \exp(X_2)),$$

more generally, for  $X_1, \dots, X_n \in \mathbf{L}((\mathcal{A}))$ ,

$$X_1 \vdash X_2 \vdash \cdots \vdash X_n = \log(\exp(X_1) \cdots \exp(X_n)).$$

The results of the Ninomiya-Ninomiya scheme are given by

**Theorem 5.2.5.** *Let  $m \geq 1$  and  $n \geq 2$ . For  $i = 1, \dots, d$ , and  $j = 1, \dots, n$ , let  $\theta_j^i$  be  $\mathbb{R}$ -valued Gaussian random variables and for  $j, j' = 1, \dots, n$ , let  $\gamma_j$  and  $\zeta_{jj'}$  be real numbers such that  $\sum_{j=1}^n \gamma_j = 1$ ,  $\mathbb{E}[\theta_j^i] = 0$ , and  $\mathbb{E}[\theta_j^i \theta_{j'}^{i'}] = \zeta_{jj'} \delta_{ii'}$  for  $i, i' = 1, \dots, d$ . We set  $\theta_j^0 = \gamma_j$ . For all family of random variables  $Z_1, \dots, Z_n$  defined by*

$$Z_j = \gamma_j \alpha_0 + \sum_{i=1}^d \theta_j^i \alpha_i \quad \text{for } j = 1, \dots, n \quad (5.2.2)$$

and such that

$$\mathbb{E}[j_m(\exp(Z_1) \cdots \exp(Z_n))] = j_m \left( \exp \left( \alpha_0 + \frac{1}{2} \sum_{i=1}^d \alpha_i^2 \right) \right). \quad (5.2.3)$$

We have that for  $p \in [1, \infty)$  and arbitrary  $g_1, \dots, g_n \in \mathcal{IS}(m)$  there exists a positive constant  $C_{m,n}$  depending on  $m$  and  $n$  such that

$$\begin{aligned} & \| \sup_{x \in \mathbb{R}^N} |g_1(\phi(\psi_s(Z_1))) \circ \cdots \circ g_n(\phi(\psi_s(Z_n)))(x) \\ & \quad - \exp(\phi(\psi_s(j_m(Z_n \vdash \cdots \vdash Z_1))))(x)| \|_{L^p} \leq C_{m,M} s^{(m+1)/2}. \end{aligned}$$

Here,  $s \in (0, 1]$ , and for any functions  $f$  and  $g$ ,  $f \circ g(x)$  denotes  $f(g(x))$ .

**Remark 5.2.6.** *It is proved in Ninomiya and Ninomiya, 2009 that for any  $m \geq 2$  there exists a set of random variable  $Z_1, \dots, Z_n$  that satisfies Equation (5.2.2) and (5.2.3). For example, when  $m = 5$  and  $n = 2$ , we can construct  $(Z_j)_{j=1,2}$  by taking*

$$\begin{aligned} \gamma_1 &= \frac{\mp \sqrt{2(2a-1)}}{2}, \quad \gamma_2 = 1 \pm \frac{\sqrt{2(2a-1)}}{2}, \quad \zeta_{11} = a \\ \zeta_{22} &= 1 + a \pm \sqrt{2(2a-1)}, \quad \zeta_{12} = -a \mp \frac{\sqrt{2(2a-1)}}{2} \end{aligned}$$

for some  $a \geq \frac{1}{2}$ .

The next step of the algorithm is to construct the integration schemes  $g_1, \dots, g_n$ . Let us consider the following ODE:

$$\frac{dY}{dt}(t, Y_0) = V(Y(t, Y_0)), \quad Y(0, Y_0) = Y_0, \quad (5.2.4)$$

where  $V \in C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$  and  $Y_0 \in \mathbb{R}^N$ . For a fixed  $M, r \in \mathbb{N}^*$ , an  $r$ th order approximation of the solution of (5.2.4) is given by the  $M$ -stage  $r$ th order Runge-Kutta method as

$$Y(Y_0; V, h) = Y_0 + h \sum_{i=1}^M b_i V(Y_i(V, h)) \quad (5.2.5)$$

where  $Y_i(V, h)$  is defined inductively by

$$Y_1(V, h) = Y_0; \quad Y_i(V, h) = Y_0 + h \sum_{j=1}^{i-1} a_{ij} V(Y_j(V, h)).$$

$h$  is a small positive real number and  $(a_{ij}, b_i)_{i,j=1,\dots,M}$  satisfy the  $r$ th order conditions defined in Section 4 of Ninomiya and Ninomiya, 2009 and we refer the reader to Butcher (2008) for their explicit constructions.

**Proposition 5.2.7.** *Let  $g(V)(Y_0) = Y(Y_0; V, 1)$  be the result of the Runge-Kutta method (5.2.5) with  $h = 1$ . Then  $g$  belongs to  $\mathcal{IS}(r)$ .*

The proof of the above Proposition can be found in Ninomiya and Ninomiya, 2009. Once these integration schemes have been constructed, one can now use the Monte Carlo or Quasi-Monte Carlo method in order to perform final computation. The order of this new approximation scheme is then given by the following Corollary (see Ninomiya and Ninomiya, 2009 for more details):

**Corollary 5.2.8.** *For  $j = 1, \dots, n$ , consider  $Z_j$  be  $\mathbf{L}(\mathcal{A})$ -valued random variables constructed as in Theorem 5.2.5. Let us define the linear operator  $Q_{(s)}$  for  $s \in (0, 1]$  as*

$$(Q_{(s)}f)(x) = \mathbb{E} [f(g_{(s)}(Z_1) \circ \dots \circ g_{(s)}(Z_n)(x))], \quad (5.2.6)$$

where  $f \in C_b^\infty(\mathbb{R}^N, \mathbb{R})$ , and  $g_{(s)}(Z_i) = g(\phi(\psi_s(Z_i)))$  for  $g \in \mathcal{IS}(m)$ , and  $\phi, \psi_s$  defined as in Definition 5.2.1. Then

$$\|P_s f - Q_{(s)}f\|_\infty \leq cs^{(m+1)/2} \|\text{grad}(f)\|_\infty \quad (5.2.7)$$

where  $c$  is a positive constant.

**Remark 5.2.9.** *It is shown in Kusuoka, 2009 that*

1. *For a Lipschitz continuous function  $f$ , the inequality (5.2.7) still holds.*
2. *For the operator  $Q_{(s)}$  defined in the above corollary, there exists a constant  $c$  such that*

$$(P_s f)(x) - (Q_{(s)}f)(x) = cs^{(m+1)/2} + \mathcal{O}(s^{(m+3)/2})$$

*holds. This implies that the Romberg extrapolation can be applied to the Ninomiya-Ninomiya algorithm.*

Following Corollary 5.2.8, we have the following theorem which presents the main result the Ninomiya-Ninomiya approximation scheme.

**Theorem 5.2.10.** *Let  $n \in \mathbb{N}^*$  be given and fixed and consider  $0 = t_0 < \dots < t_n = T$  a partition of  $[0, T]$  and let  $s_k = t_k - t_{k-1}$ ,  $k = 1, \dots, n$ .*

*Then*

$$\|P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f\|_\infty \leq \frac{C_T}{n^{(m-1)/2}} \|f\|_\infty,$$

*where  $C_T$  is a positive constant, each  $Q_{(s_i)} f$  is defined as in Equation (5.2.6) with  $g \in \mathcal{IS}(m)$ .*

### 5.2.3 Application

In this Subsection, we illustrate the implementation of the algorithm proposed in the previous Section. We apply the Ninomiya-Ninomiya weak approximation scheme to the pricing of the Asian call option presented in Subsection 5.1.2.1. Following Equation (5.1.7), the SDE we consider here is given in Stratonovich form as

$$X(t, x) = x + \sum_{i=0}^2 \int_0^t V_i(X(s, x)) \circ dB^i(s), \quad (5.2.8)$$

with

$$\begin{aligned} V_0(X) &= \left( X_1 \left( \mu - \frac{1}{2} X_2 - \frac{1}{4} \rho \sigma \right), \kappa(\theta - X_2) - \frac{1}{4} \sigma^2, X_1 \right)', \\ V_1(X) &= \left( X_1 \sqrt{X_2}, \rho \sigma \sqrt{X_2}, 0 \right)', \\ V_2(X) &= \left( 0, \sigma \sqrt{1 - \rho^2} \sqrt{X_2}, 0 \right)', \end{aligned}$$

where

$$X = (X_1, X_2, X_3)' = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \quad \text{and} \quad x = (x_1, x_2, x_3)'.$$

From Remark 5.2.6 and taking  $a = \frac{3}{4}$ , in order to implement the algorithm, we choose the following quantities:

$$\gamma_1 = \frac{1}{2}, \gamma_2 = \frac{1}{2}, \zeta_{11} = \frac{3}{4}, \zeta_{12} = -\frac{1}{4}, \text{ and } \zeta_{22} = \frac{11}{4}.$$

We construct the family  $(\theta_j^i)_{i,j=1,2}$  of  $\mathbb{R}$ -valued Gaussian random variables which verifies

$$E[\theta_j^i] = 0, \quad \text{and} \quad E[\theta_j^i \theta_{j'}^{i'}] = \zeta_{jj'} \delta_{ii'} \quad (5.2.9)$$

as follows: Let  $\eta_1, \eta_2, \eta_3, \eta_4$  be four independent standard normal random variables and set

$$\theta_1^1 = \frac{\sqrt{3}}{2} \eta_1, \theta_2^1 = -\frac{\sqrt{3}}{6} \eta_1 + \frac{2\sqrt{6}}{3} \eta_2, \theta_1^2 = -\frac{\sqrt{3}}{2} \eta_3, \theta_2^2 = \frac{\sqrt{3}}{6} \eta_3 - \frac{2\sqrt{6}}{3} \eta_4.$$

For details about this construction see Section A.1 in Appendix.

It is not difficult to verify that the above random variables satisfy (5.2.9) as desired. The next step is to now derive random variables  $Z_1, Z_2$  as defined in Equation (5.2.2),

$$Z_1 = \frac{1}{2}\alpha_0 + \frac{\sqrt{3}}{2}\eta_1\alpha_1 - \frac{\sqrt{3}}{2}\eta_3\alpha_2, \quad (5.2.10)$$

$$Z_2 = \frac{1}{2}\alpha_0 - \frac{\sqrt{3}}{6}\eta_1\alpha_1 + \frac{2\sqrt{6}}{3}\eta_2\alpha_1 + \frac{\sqrt{3}}{6}\eta_3\alpha_2 - \frac{2\sqrt{6}}{3}\eta_4\alpha_2, \quad (5.2.11)$$

where  $(\alpha_0, \alpha_1, \alpha_2)$  is a multi-index. We compute the functions  $\phi\psi_s(Z_1)$  and  $\phi\psi_s(Z_2)$ , for  $s > 0$ . Set

$$\begin{aligned} \phi_s^1 &= \phi\psi_s(Z_1), \\ \phi_s^2 &= \phi\psi_s(Z_2), \end{aligned}$$

by the definition of  $\psi_s$ , we have that

$$\begin{aligned} \psi_s(Z_1) &= \frac{1}{2}s\alpha_0 + \frac{\sqrt{3}}{2}\eta_1\sqrt{s}\alpha_1 - \frac{\sqrt{3}}{2}\eta_3\sqrt{s}\alpha_2, \\ \psi_s(Z_2) &= \frac{1}{2}s\alpha_0 - \frac{\sqrt{3}}{6}\eta_1\sqrt{s}\alpha_1 + \frac{2\sqrt{6}}{3}\eta_2\sqrt{s}\alpha_1 + \frac{\sqrt{3}}{6}\eta_3\sqrt{s}\alpha_2 - \frac{2\sqrt{6}}{3}\eta_4\sqrt{s}\alpha_2 \end{aligned}$$

and using the definition of  $\phi$  we obtain

$$\begin{aligned} \phi_s^1 &= \frac{1}{2}sV_0 + \frac{\sqrt{3}}{2}\eta_1\sqrt{s}V_1 - \frac{\sqrt{3}}{2}\eta_3\sqrt{s}V_2, \\ \phi_s^2 &= \frac{1}{2}sV_0 - \frac{\sqrt{3}}{6}\eta_1\sqrt{s}V_1 + \frac{2\sqrt{6}}{3}\eta_2\sqrt{s}V_1 + \frac{\sqrt{3}}{6}\eta_3\sqrt{s}V_2 - \frac{2\sqrt{6}}{3}\eta_4\sqrt{s}V_2. \end{aligned}$$

To compute  $g_{(s)}(Z_1) = g(\phi_s^1)$  and  $g_{(s)}(Z_2) = g(\phi_s^2)$ , we consider the systems of ODE

$$\frac{dY}{dt}(t, Y_0) = \phi_s^k(Y(t, Y_0)), \quad Y(0, Y_0) = Y_0, \quad k = 1, 2 \quad s > 0 \quad (5.2.12)$$

and using the Runge-Kutta method presented in Equation (5.2.5) with  $h = 1$  we obtain

$$\begin{aligned} Y_i(\phi_s^k, 1) &= Y_0 + \sum_{j=1}^M a_{ij}\phi_s^k(Y_j(\phi_s^k, 1)), \\ Y(Y_0; \phi_s^k, 1) &= Y_0 + \sum_{i=1}^M b_i\phi_s^k(Y_i(\phi_s^k, 1)), \end{aligned}$$

and by setting  $g(\phi_s^k)(Y_0) = Y(Y_0; \phi_s^k, 1)$ . We have by Proposition 5.2.7 that  $g(\phi_s^k)(Y_0) = Y(Y_0; \phi_s^k, 1)$ , so

$$g_{(s)}(Z_1) \circ g_{(s)}(Z_2)(x) = g(\phi_s^1)(g(\phi_s^2)(x)) = Y(Y(x; \phi_s^2, 1); \phi_s^1, 1).$$



We then conclude by setting  $g_s(x) = g_{(s)}(Z_1) \circ g_{(s)}(Z_2)(x)$  that

$$Q_{(s)}f(x) = \mathbb{E}[f(g_s(x))]$$

where the payoff function  $f(x) = \max(0, \frac{x_3}{T} - K)$ . This implies

$$Q_{(s_n)}Q_{(s_{n-1})} \cdots Q_{(s_1)}f(x) = \mathbb{E}[f(g_{s_n}(g_{s_{n-1}} \cdots g_{s_1}(x)))] .$$

To obtain integration schemes of order at least 5 we consider the Runge-Kutta method with  $r = 5$  and then the Ninomiya-Ninomiya new approximation scheme is of order 2. For the Romberg extrapolation we take  $r = 7$ .

The coefficients  $(a_{ij})_{i,j}$  and  $(b_j)_j$  for the fifth and the seventh order are taken from (Butcher, 2008) as:

The fifth order coefficients

$$\begin{aligned} a_{21} &= \frac{1}{4}, & a_{31} &= \frac{1}{8}, & a_{32} &= \frac{1}{8}, & a_{43} &= \frac{1}{2}, & a_{51} &= \frac{3}{16}, & a_{52} &= -\frac{3}{8}, & a_{53} &= \frac{3}{8} \\ a_{54} &= \frac{9}{16}, & a_{61} &= -\frac{3}{7}, & a_{62} &= \frac{8}{7}, & a_{63} &= \frac{6}{7}, & a_{64} &= -\frac{12}{7}, & a_{65} &= \frac{8}{7} \\ a_{ij} &= 0 \quad \text{otherwise,} \\ b &= \left( \frac{7}{90}, \quad 0, \quad \frac{32}{90}, \quad \frac{12}{90}, \quad \frac{32}{90}, \quad \frac{7}{90} \right). \end{aligned}$$

The seventh order coefficients

$$\begin{aligned} a_{21} &= \frac{1}{6}, & a_{32} &= \frac{1}{3}, & a_{41} &= \frac{1}{8}, & a_{43} &= \frac{3}{8}, & a_{51} &= \frac{148}{1331}, & a_{53} &= \frac{150}{1331} \\ a_{54} &= -\frac{56}{1331}, & a_{61} &= -\frac{404}{243}, & a_{63} &= -\frac{170}{27}, & a_{64} &= \frac{4024}{1701}, & a_{65} &= \frac{10648}{1701} \\ a_{71} &= \frac{2466}{2401}, & a_{73} &= \frac{1242}{343}, & a_{74} &= -\frac{19176}{16807}, & a_{75} &= -\frac{51909}{16807}, & a_{76} &= \frac{1053}{2401} \\ a_{81} &= \frac{5}{154}, & a_{84} &= \frac{96}{539}, & a_{85} &= -\frac{1815}{20384}, & a_{86} &= -\frac{405}{2464}, & a_{87} &= \frac{49}{1144} \\ a_{91} &= -\frac{113}{32}, & a_{93} &= -\frac{195}{22}, & a_{94} &= \frac{32}{7}, & a_{95} &= \frac{29403}{3584}, & a_{96} &= -\frac{729}{512} \\ a_{97} &= \frac{1029}{1408}, & a_{98} &= \frac{21}{16}, & a_{ij} &= 0 \quad \text{otherwise,} \\ b &= \left( 0, \quad 0, \quad 0, \quad \frac{32}{105}, \quad \frac{1771561}{6289920}, \quad \frac{243}{1560}, \quad \frac{16807}{74880}, \quad \frac{77}{1440}, \quad \frac{11}{70} \right). \end{aligned}$$

### 5.3 Conclusion

In this thesis, our main concern has been the presentation of recently developed higher order schemes for weak approximation of SDEs. Our main concern has been the presentation of recently developed higher order schemes.

We have started with the Kusuoka approximation scheme which can be viewed as the starting point of all the presented approximation schemes. It is based on the concept of a  $m$ -moment similar family of random variables. Remarkable improvements in the speed and accuracy have been observed when using the Kusuoka approximation scheme instead of the traditional Euler-Maruyama method. This scheme was intensively developed by Lyons and Victoir. The main results of the Lyons-Victoir approximation scheme, namely cubature on Wiener space have also been explored. The construction of cubature formulas on Wiener space has appeared to be the main problem in the implementation of the cubature on Wiener space method. We have presented the main results of the Ninomiya-Victoir and Ninomiya-Ninomiya approximation schemes. These two schemes are of second order and can be viewed as extensions and applications of the Kusuoka and cubature on Wiener space methods. From the implementation of the Ninomiya-Victoir method, we have observed that, compared to the Euler-Maruyama method, the Ninomiya-Victoir algorithm is much faster, needs fewer number of discretization points and is more accurate. Despite the fact that the simulation of the Ninomiya-Ninomiya algorithm has not been performed to compare the two algorithms, we have observed that the Ninomiya-Ninomiya implementation method is complete and distinct mainly because the computation of  $\exp(Z_j)$  is part of the algorithm whereas it is not the case in the Ninomiya-Victoir method.

We have only focused this work on the presentation of general ideas and algorithms of these approximation schemes. There is still a lot to investigate such as:

- (1) Improving these approximation schemes in order to obtain schemes of order greater than 2.
- (2) Adapting these results to the case where the function  $f$  does not satisfy even Lipschitz continuity or the vector fields in consideration are not smooth.

Another area of particular interest for future work is the extension of applications by adapting these algorithms to other financial problems such as the computation of Greeks or the pricing of risk.

# Appendix A

## A.1 Construction of the Family of $\mathbb{R}$ -valued Gaussian Random Variables

In this section, we construct the family  $(\theta_j^i)_{i,j}$  for  $i = 1, 2$  and  $j = 1, 2$  of  $\mathbb{R}$ -valued Gaussian random variables defined in Theorem 5.2.5. We fix

$$\gamma_1 = \frac{1}{2}, \quad \gamma_2 = \frac{1}{2}, \quad \zeta_{11} = \frac{3}{4}, \quad \zeta_{12} = -\frac{1}{4}, \quad \zeta_{22} = \frac{11}{4}.$$

Let  $\eta_1, \eta_2, \eta_3, \eta_4$  be four independent standard normal random variables and

$$\theta_1^1 = a_1\eta_1, \quad \theta_2^1 = a_2\eta_1 + a_3\eta_2, \quad \theta_1^2 = a_4\eta_3, \quad \theta_2^2 = a_5\eta_3 + a_6\eta_4.$$

$\theta_j^i$  satisfy the conditions

$$E[\theta_j^i] = 0, \quad \text{and} \quad E[\theta_j^i \theta_{j'}^{i'}] = \zeta_{jj'} \delta_{ii'}$$

is equivalent to  $a_1, a_2, a_3, a_4, a_5$  and  $a_6$  satisfy

$$a_1^2 = a_4^2 = \zeta_{11} \tag{A.1.1}$$

$$a_2^2 + a_3^2 = a_5^2 + a_6^2 = \zeta_{22} \tag{A.1.2}$$

$$a_1 a_2 = a_4 a_5 = \zeta_{12}. \tag{A.1.3}$$

Equation (A.1.1) implies  $a_1, a_4 \in \left\{ \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \right\}$ , we then fix  $a_1 = \frac{\sqrt{3}}{2}$  and  $a_4 = -\frac{\sqrt{3}}{2}$ . We use equation (A.1.3) to compute  $a_2 = -\frac{\sqrt{3}}{6}$  and  $a_5 = \frac{\sqrt{3}}{6}$ . Equation (A.1.2) gives  $a_3 = \frac{2\sqrt{6}}{3}$  and  $a_6 = -\frac{2\sqrt{6}}{3}$  so

$$\begin{aligned} \theta_1^1 &= \frac{\sqrt{3}}{2} \eta_1, & \theta_2^1 &= -\frac{\sqrt{3}}{6} \eta_1 + \frac{2\sqrt{6}}{3} \eta_2, \\ \theta_1^2 &= -\frac{\sqrt{3}}{2} \eta_3, & \theta_2^2 &= \frac{\sqrt{3}}{6} \eta_3 - \frac{2\sqrt{6}}{3} \eta_4. \end{aligned}$$

## A.2 Python Codes for Simulations

### A.2.1 Code for Kusuoka Approximation

```
%\begin{verbatim}breakchar{\char'\%}
"""
This program computes the price of an asian option using Kusuoka approximation.
"""

from __future__ import division
import math
from scipy import *
from scipy.stats import *
from random import *

# we implement random variables needed for kusuoka scheme
w = [0,sqrt(2+sqrt(2)), -sqrt(2+sqrt(2)), sqrt(2-sqrt(2)), -sqrt(2-sqrt(2))]
#we compute the probability list
P = [1/2, 1/8, 1/8, 1/8, 1/8]

def ar(a,b):
    """this function returns the array([a,b])
    """
    c = array([a,b])
    return c

def funcg(s,etha,x):
    """this function returns G(s, etha, x) we have in the formula where x = [x_1,x_2]
    """

    G= ar(x[0],x[1]) + sqrt(s)*etha*ar(sigma*x[0],0) + s*ar(mu*x[0],x[0]) +
    s**2*(1/12 - etha**2/18)*ar(0,sigma**2*x[0]) + s**2*etha**2/18*ar(0,x[0]*sigma**2) +
    sqrt(s**5)*((-etha+etha**3)/32)*ar(0,x[0]*sigma**3) + sqrt(s**5)*(etha**3/96-etha/32)
    *ar(0,-x[0]*sigma**3) + s*etha**2/2*ar(x[0]*sigma**2,0) + sqrt(s**3)*etha/2
    *ar(mu*sigma*x[0],sigma*x[0]) + sqrt(s**3)*etha/2*ar(mu*sigma*x[0],0) +
    sqrt(s**5)*etha**3/36*ar(0,x[0]*sigma**3) + sqrt(s**5)*(etha/24 - etha**3/36)
    *ar(0,x[0]*sigma**3) + 1/6*etha**3*sqrt(s**3)*ar(x[0]*sigma**3,0) +
    s**2*etha**2/6*ar(mu*x[0]*sigma**2,x[0]*sigma**2) + s**2*etha**2/3*ar(mu*x[0]
    *sigma**2,0) + sqrt(s**5)*etha/3*ar(mu**2*x[0]*sigma,mu*sigma*x[0]) +
    sqrt(s**5)*etha/6*ar(mu**2*x[0]*sigma,0) + s**2*etha**4/24*ar(x[0]*sigma**4,0) +
    sqrt(s**5)*etha**3/24*ar(mu*x[0]*sigma**3,sigma**3*x[0]) + 1/8*sqrt(s**5)
    *etha**3*ar(mu*x[0]*sigma**3,0) + 1/120*sqrt(s**5)*etha**5*ar(sigma**5*x[0],0)

    #print "G=", G
    return G
```

```

def list_s(n):
    """ this function returns the partitioning of the interval [0,T]
    t_k = k^gamma * n^-gamma *T and s_k = t_k - t_{k-1}
    """
    s = zeros(n)
    t = zeros(n+1)
    t[0] = 0
    for j in range(1,n+1):
        t[j] = ((j/n)**gamma)*T
    s[j-1] = t[j] - t[j-1]
    return s

def som(l,x):
    """this function returns the computation of G(s[0],w[l[0]],G(s[1],w[l[1]],G(...,
    G(s[n-1],w[l[n-1]],x)...))), x is the initial value into consideration.
    """
    s = list_s(n)
    comp = x
    for i in range(n-1,-1,-1):
        comp = funcg(s[i],w[l[i]],comp)
    return comp

def prod(l):
    """this function returns the value of the product we have to compute, that is:
    pro_j=1^n p[l[j]]
    """
    pro = 1
    for k in range(n):
        i = int(l[k])
        pro = pro*P[i]
    return pro

def maxs(x):
    """this function returns the max between x_2/T-k and 0, x here a list off
    two elements
    """
    c = x[1]/T
    f = max(c-K, 0)
    return f

def loop(n,N):
    """
    Loop over base N number of length n

```

```

00..0 -> NN..N
"""
# use array k
k=zeros(n)
M = zeros([N**n,n])

for i in range(N**n): # max size of baseN number of length n
M[i] = k
#print "M= ", M
#print k # do your calculation with k here
# Increment by one (modulo N)
k[n-1] = (k[n-1]+1) % N
# carry bits
for j in range(n-1,-1,-1): # right-to-left
if k[j] == 0:
k[j-1] = (k[j-1]+1) % N
else:
break
return M

# main =====
if __name__ == "__main__":
# constants
r = 0.05
sigma = 0.3
mu = r-(sigma**2)/2
K = 1.05
T = 0.25
x_1 = 1
x_2 = 0
X = array([x_1,x_2])
# we discretize the interval [0,T]
gamma = 2
n = 8 # for the loop function and also the number of discretization points
N = 5 # for the loop function
s = list_s(n)

Sum = 0
M = loop(n,N) # we call for the loop function
ni = len(M)
#print ni
for j in range(ni):
l = list(M[j])
#print l

```

```

for i in range(len(l)):
    b = int(l[i])
    l[i] = b
#print l
comp = som(l,X)
pro = prod(l)
com = maxs(comp)
Sum = Sum + pro*com
#print Sum
ku_price = exp(-r*T)*Sum
print ku_price .

```

### A.2.2 Code for Ninomiya-Victoir Algorithm Applied to Option Pricing

```

"""
This program computes the price of an asian option under heston volatility
using Ninomiya-victoir algorithm.
"""

from __future__ import division
import math
from scipy import *
from scipy.stats import *
from random import *

def expV0(s,l):
    """this function returns the value of exp(sV0)(l), l=[y_1,y_2,y_3].
    """
    v = zeros(3)
    v[0] = l[0]*exp(B*s +(exp(-alpha*s)-1)*(l[1]-J)/(2*alpha))
    v[1] = (l[1] -J)*exp(-alpha*s) + J
    v[2] = l[2] + (exp(A*s)-1)*l[0]/A
    return v
def expV1(s,l):
    """this function returns the value of exp(sV0)(l), l=[y_1,y_2,y_3].
    """
    v = zeros(3)
    v[0] = l[0]*exp(s*sqrt(l[1]) + (s**2)*rho*beta/4)
    v[1] = (rho*beta*s/2 + sqrt(l[1]))**2
    v[2] = l[2]
    return v
def expV2(s,l):
    """this function returns the value of exp(sV0)(l), l=[y_1,y_2,y_3].
    """

```

```

v = zeros(3)
v[0] = l[0]
v[1] = (sqrt(1-(rho**2))*beta*s/2 + sqrt(l[1]))**2
v[2] = l[2]
return v
# main =====
if __name__ == "__main__":
# constants
mu = 0.05
beta = 0.1
rho = 0.7
theta = 0.09
alpha = 2.0
K = 1.05
T = 1

# we implement the algorithm.
d = 2 #input("enter the dimension of the brownian motion.")
n = 13 #input("enter the numbers of partition")
y_1 = 1.0 #input("enter the first initial condition")
y_2 = 0.09 #input("enter the second initial condition")
y_3 = 0.0 #input("enter the third initial condition")
# here a some useful coefficients for the the computation
J = theta - beta**2/(4*alpha)
A = mu - y_2/2 - rho*beta/4
B = mu - J/2 - rho*beta/4
X = [y_1,y_2, y_3]
monte = 40000 # numbers of sample points used for the monte carlo simulation
k = monte

Sum1 = 0
Sum2 = 0
l3 = zeros(5)
for p in range(5):
Sum1 = 0
for j in range(0,k):
X = [y_1,y_2, y_3]
for i in range(1,n+1):
# here we compute X considering the new approximation scheme
#we ask for bernoulli and normal random variables
ber = randint(0,1) #this choose randomlly 0 or 1
nor = zeros(d)
for l in range(0,d):
nor[l] = gauss(0,1) # this choose a normal random variable.

```



```

if ber == 0:
# we compute exp(sV)
    X = expV0(1/(2*n),expV2(nor[1]/sqrt(n),expV1(nor[0]/sqrt(n),expV0(1/(2*n),X))))
else:
# we compute exp(sV)
    X = expV0(1/(2*n),expV1(nor[0]/sqrt(n),expV2(nor[1]/sqrt(n),expV0(1/(2*n),X))))
# we now compute the payoff of this call option.
#print X[2]
Average1 = X[2]/T
# here we compute the Payoff for the Ninomiya-victoir scheme
if Average1>K:
Pf1 = Average1 - K # here we denote the payoff by Pf
else:
Pf1 = 0

Sum1 +=Pf1 # adding the previous Sum with the new value of the payoff

#print Sum1
# we compute the result (Res) which is the average of the Sum and which
gives the expectation value using monte carlo
Res1 = Sum1/k
#print 'Res1=', Res1
#print 'k=', k
l3[p] = Res1
Sum2 = Sum2+Res1
tot = Sum2/5
print "tot = ", tot
error = tot - 0.0604194813
print "error with n = ", n, "= ", error

```

### A.2.3 Code for Euler-Maruyama Method Applied to Option Pricing

```

"""
This program computes the price of an asian option under heston volatility
using Euler-maruyama algorithm.
"""

from __future__ import division
import math
from scipy import *
from scipy.stats import *
from random import *

```

```

# main =====
if __name__ == "__main__":
# constants
mu = 0.05
beta = 0.1
rho = 0.7
theta = 0.09
alpha = 2.0
K = 1.05
T = 1

# we implement the algorithm.
d = 2 #input("enter the dimension of the brownian motion.")
n = 2000 #input("enter the sample points to use")
y_1 = 1.0 #input("enter the first initial condition")
y_2 = 0.09 #input("enter the second initial condition")
y_3 = 0.0 #input("enter the third initial condition")
Y = [y_1,y_2, y_3]
monte = 40000 # numbers of sample points used for the monte carlo simulation
k = monte
Sum2 = 0
for j in range(0,k):
Y = [y_1,y_2, y_3]
for i in range(1,n+1):
# here we compute X considering the new approximation scheme
#we ask for bernoulli and normal random variables
nor = zeros(d)
for l in range(0,d):
nor[l] = gauss(0,1) # this choose a normal random variable.
# here we compute X considering the euler-maruyama scheme
Y1 = Y[0] + mu*Y[0]/n + Y[0]*sqrt(Y[1])*sqrt(1/n)*nor[0]
Y2 = Y[1] + alpha*(theta-Y[1])/n + beta*sqrt(Y[1])*(rho*sqrt(1/n)*nor[0] +
sqrt(1-(rho**2))*sqrt(1/n)*nor[1])
Y3 = Y[2] + Y[0]/n
Y[0] = Y1
Y[1] = Y2
Y[2] = Y3
# we now compute the payoff of this call option.
#print Y[2]
Average2 = Y[2]/T
#here we compute the Payoff for the Euler-Maruyama scheme
if Average2>K:
Pf2 = Average2 - K # here we denote the payoff by Pf
else:

```

```
Pf2 = 0

Sum2 +=Pf2 # adding the previous Sum with the new value of the payoff
# we now do the computation for the negative part
#print sum2
# we compute the result (Res) which is the average of the Sum and which
gives the expectation value using monte carlo
Res2 = Sum2/k
print 'Res2 with n= ', n, ' = ', Res2
error = Res2-0.0604194813
print "error = ", error
```

#### A.2.4 Codes for Ninomiya-Victoir Method Applied to Optimal Portfolio Strategies

#### A.2.5 Codes for Euler-Maruyama Method Applied to Optimal Portfolio Strategies

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